Some properties of closed hypersurfaces of small entropy and the topology of hypersurfaces through singularities of mean curvature flow

by
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Abstract

We record in this thesis three results concerning entropy and singularities in mean curvature flow (MCF).

The first result is a stability result of round spheres under small-entropy perturbation. The round spheres are minimizer of the entropy functional and we show that in all dimensions a closed hypersurface must be close to a round sphere in Hausdorff distance if the entropy is close to that of a round sphere. This generalizes a result of Bernstein-Wang in dimension 2.

The second result gives a sharp entropy lower bound for disconnection to happen in mean curvature flow of hypersurfaces in $\mathbb{R}^4$. And it's related to the first result in that it sharpens the condition of a uniform continuity estimate of Hausdorff distance over time. The non-sharp version of this uniform continuity was used as a key lemma in the proof of the first result. This second result is joint work with J. Benstein.

The third result is a rigidity result in the singularity models of mean curvature flow. Self-shrinkers are singularity models in mean curvature flow by Huisken's monotonicity formula. And by using techniques from minimal surfaces, we showed that a self-shrinking torus must be unknotted. This third result is joint work with A. Mramor.

READERS: Professor Jacob Bernstein (Advisor), Joel Spruck,

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I dedicate this dissertation to my parents. Thank you for the love and support in my life.

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Introduction

The mean curvature flow (MCF) is a natural geometric evolution of hypersurfaces (or more general submanifolds). At each point of the evolving hypersurfaces, the normal component of the velocity vector is equal to the mean curvature vector. It is the negative gradient flow of the volume, so hypersurfaces will move in the direction that volume decreases most rapidly. The first mathematical study of mean curvature flow is due to Brakke [7] in the context of geometric measure theory.

MCF has been an important model in science and engineering, for example it is the model of some physical phenomena such as formation of grain boundaries [31] in annealing metals and the evolution of soap films to equilibrium state and it has found application in image processing. Within the area of pure mathematics, MCF are also have application and are related to other questions in geometric topology and general relativity.

A key feature of MCF equation is that singularities are unavoidable: For any initial data that is a closed hypersurface, the evolution will always develop a singularity and become extinct in finite time by the parabolic maximum principle. A central theme in studying the flow is to understand the long time behavior of the flow: the formation and classification of singularities, and the evolution of the flow as it passes through singularities.

We are still far away from getting a complete classification of singularities of mean curvature flow even for the 2-dimensional surfaces in $\mathbb{R}^3$. In a profound work of Colding and Minicozzi [10], they introduced a dynamical point of view to study singularities of mean curvature flow. To do this they defined an important entropy functional, which is a monotonic quantity for mean curvature flow. In their dynamical picture, the generic singularities of the mean curvature flow must be entropy stable, otherwise they can be perturbed away. Moreover, Colding and Minicozzi were able to classify all the entropy stable singularities, namely those that cannot be avoided by generic perturbations.

The Colding-Minicozzi entropy functional is a measure of complexity of hypersurfaces and the singularity models. It is natural to ask what are the minimizer of entropy in the class of closed hypersurfaces and, due to relation to MCF, of singularity models in MCF. Furthermore, one is interested in stability of the minimizer in some sense. In [11], Colding, Ilmanen, Minicozzi and White showed the round sphere has the lowest entropy of any closed singularity model. In [3], Bernstein-Wang showed that the round spheres $\mathbb{S}^n$ uniquely minimize the entropy (modulo dilations and rigid motions) among closed hypersurfaces in $\mathbb{R}^{n+1}$ for $2 \leq n \leq 6$, giving an affirmative answer to a conjecture made by Colding-Ilmanen-Minicozzi-White in [11]. Later, Zhu [42] extended the result to all higher dimensions. In [2], Bernstein-Wang further showed that, for $n = 2$, the round sphere is Hausdorff stable under small perturbations of entropy. The first result of this thesis is to record the Hausdorff stability of round spheres under small perturbation of the entropy in all dimensions.

In the time interval that the mean curvature flow is defined in the classical sense, the surface will evolve smoothly and there are no topological change. In the case when singularities occur (which is always the case in finite as explained previously), there are various ways to extend the flow through singularities (either through weak solutions of PDE or by some topological surgery). Among others, there is a canonical set theoretic weak mean curvature flow that persists through singularities called the level-set flow (see Chen-Giga-Goto [9] and Evans-Spruck [14, 15, 16, 17]).

When $n = 1$, it follows from Gage-Hamilton [21] and Grayson [23] that the flow disappears when it becomes singular. In particular, the flow remains connected until it disappears. In contrast, when $n > 1$, non-degenerate neck-pinch examples show that there are flows that become singular without disappearing. In these examples, the level set flow disconnects after the neck-pinch singularity (see figure below). In the mean convex case, White [41] proved a sharp entropy bound for the k-th homotopy group of the complement to die out in the level-set flow. In [5], the first author and L. Wang showed that, when $n = 2$ and the entropy of the initial surface is small enough, then the flow also disappears at its first singularity. In joint work with Bernstein [6], we show that when $n = 3$ and the initial hypersurface is closed, connected and with entropy below that of a round cylinder, then even if the flow forms a singularity before it disappears, its level set flow remains connected until its extinction time. The entropy bound is sharp because the neck-pinch singularity model has entropy equal to that of the round cylinder.

As mentioned above, singularity models of mean curvature can be very complicated geometrically and topologically and there are lots of examples. Rigidity of certain singularities are important

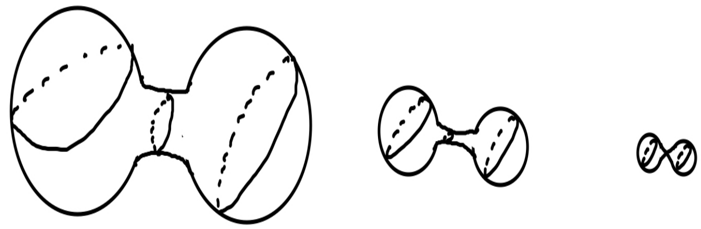

Figure 1.1: Neckpinch singularity

questions in the classification of singularities. Huisken [24] showed that under a uniform curvature bound and polynomial volume growth condition, the only mean convex singularity models are either round spheres or round cylinders. Colding and Minicozzi [10] were able to remove the condition on polynomial volume growth and gives a rigidity of the round spheres and cylinders in the mean convex class. A topological rigidity without any curvature assumption is obtained by Brendle [8], he gives a complete classification of 2-dimensional singularity models with genus 0 in $\mathbb{R}^3$. Inspired by an analogous result on the topological rigidity of minimal surfaces, we showed in [30] that closed 2-dimensional singularity models in $\mathbb{R}^3$ must be topological standard, in particular, closed singularity models with genus 1 are unknotted. This is joint work with Mramor.

2

Notation and Background

2.1 Notation

We denote by $B_R^{n+1}(x_0)$ and $\bar{B}_R^{n+1}(x_0)$ the open ball and closed ball in $\mathbb{R}^{n+1}$ with radius R and centered at x_0 respectively. We omit the super-script when the dimension is clear from context. Let $T_r(K) = \bigcup_{x \in K} B_r(x)$ be the r-tubular neighborhood of K.

Given two compact subsets $X, Y \subset \mathbb{R}^{n+1}$, the Hausdorff distance $\mathrm{dist}_H(X, Y)$ between X and Y is defined by

$$\mathrm{dist}_H(X, Y) = \inf\{r > 0 : X \subset \cup_{x \in Y} \bar{B}_r(x) \; and \; Y \subset \cup_{x \in X} \bar{B}_r(x)\}$$

.

For any $\rho > 0$, $x_0 \in \mathbb{R}^{n+1}$ and $\Omega \subset \mathbb{R}^{n+1}$, we donote by

$$\Omega - x_0 = \{x \in \mathbb{R}^{n+1} : x + x_0 \in \Omega\}$$

$$\rho\Omega = \{\rho x : x \in \Omega\}$$

Following the notations in [25], we denote by

- $\mathbf{M}(\mathbb{R}^{n+1}) = \{\mu\text{: } \mu \text{ is a Radon measure on } \mathbb{R}^{n+1}\}$
- $\mathbf{IM}_k(\mathbb{R}^{n+1}) = \{\mu\text{: } \mu \text{ is an integer k-rectifiable Radon measure on } \mathbb{R}^{n+1}\}$
- $\mathbf{I}_k(\mathbb{R}^{n+1}) = \{\text{T: T is an integeral k-current on } \mathbb{R}^{n+1}\}$
- $\mathbf{IV}_k(\mathbb{R}^{n+1}) = \{\text{V: V is an integer k-rectifiable varifold on } \mathbb{R}^{n+1}\}$

$\mathbf{M}(\mathbb{R}^{n+1})$, $\mathbf{IM}_k(\mathbb{R}^{n+1})$ and $\mathbf{IV}_k(\mathbb{R}^{n+1})$ are equipped with corresponding weak* topologies. $\mathbf{I}_k(\mathbb{R}^{n+1})$

is equipped with the flat topology. See [25] section 1 for details of the topologies and corresponding compactness theorems.

There are natural maps

$$V_1 : \mathbf{IM}_k(\mathbb{R}^{n+1}) \to \mathbf{IV}_k(\mathbb{R}^{n+1})$$
$$V_2 : \mathbf{I}_k(\mathbb{R}^{n+1}) \to \mathbf{IV}_k(\mathbb{R}^{n+1})$$
$$\mu_1 : \mathbf{I}_k(\mathbb{R}^{n+1}) \to \mathbf{IM}_k(\mathbb{R}^{n+1})$$
$$\mu_2 : \mathbf{IV}_k(\mathbb{R}^{n+1}) \to \mathbf{IM}_k(\mathbb{R}^{n+1})$$

Of the above maps, only μ_2 is continuous. We use the following notations for convenience:

$$V_1(\mu) = V(\mu) = V_\mu$$
$$V_2(T) = V(T) = V_T$$
$$\mu_1(T) = \mu(T) = \mu_T$$
$$\mu_2(V) = \mu(V) = \mu_V$$

Following definitions in [40], an integral current $T \in \mathbf{I}_k(\mathbb{R}^{n+1})$ and an integer k-rectifiable (integral) varifold $V \in \mathbf{IV}_k(\mathbb{R}^{n+1})$ are said to be compatible if $V = V(T) + 2W$ for some integral varifold $W \in \mathbf{IV}_k(\mathbb{R}^{n+1})$.

If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a hypersurface, we denote by $\mu_\Sigma = \mathcal{H}^n \lfloor \Sigma \in \mathbf{IM}_n(\mathbb{R}^{n+1})$.

We also define the set of self-shrinking measures on $\mathbb{R}^{n+1}$ by

$$\mathcal{SM}_n = \{\mu \in \mathbf{IM}_n(\mathbb{R}^{n+1}) : V_\mu \text{ is stationary for Gaussian area F}\}$$

Denote by

$$\mathcal{CSM}_n = \{\mu \in \mathcal{SM}_n : \mu \text{ has compact support}\}$$

Further, given $\Lambda > 0$, set

$$\mathcal{SM}_n(\Lambda) = \{\mu \in \mathcal{SM}_n : \lambda(\mu) < \Lambda\}$$
$$\mathcal{CSM}_n(\Lambda) = \mathcal{CSM}_n \cap \mathcal{SM}_n(\Lambda)$$

The generic singularities are either round spheres or generalized round cylinders by [10] and we denote their entropy by

$$\Lambda_k = \lambda(\mathbb{S}^k) = \lambda(\mathbb{S}^k \times \mathbb{R}^{n-k}) = \lambda(\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k})$$

By [10], the entropy of a self-shrinker is equal to the value of Gaussian area functional F. Stone [34] computed the Gaussian area functional, and therefore the entropy, for generalized cylinders and found that they are monotonic decreasing in dimension

$$\Lambda_1 = \sqrt{\frac{2\pi}{e}} \approx 1.52 > \Lambda_2 = \frac{4}{e} \approx 1.48 > \Lambda_3 > ... > \Lambda_n \to \sqrt{2}.$$

2.2 Mean curvature flow

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $n+1$-dimensional Euclidean space. The mean curvature vector is defined by

$$\mathbf{H} = -\mathrm{div}(\mathbf{n})\mathbf{n} = -H\mathbf{n},$$

where H is called the scalar mean curvature and $\mathbf{n}$ is a choice unit normal vector field.

A family of hypersurfaces Σ_t are said to be evolving by mean curvature flow if the normal component of the velocity vector is equal to the mean curvature vector, namely

$$\frac{d\mathbf{x}_\Sigma}{dt} = \mathbf{H}_\Sigma.$$

Since the mean curvature vector in the Euclidean space is equal to $\mathbf{H}_\Sigma = \Delta_\Sigma \mathbf{x}$, the equation of mean curvature flow is a nonlinear heat equation

$$\frac{d\mathbf{x}_\Sigma}{dt} = \Delta_\Sigma \mathbf{x},$$

where the nonlinearity comes from the fact that the Laplacian is the one restricted to the submanifold metric and it's changing in time. As a parabolic equation, the mean curvature flow has a parabolic rescaling and translation property. Namely, if $\{\Sigma_t\}_{t\in\mathbb{R}}$ is a mean curvature flow in $\mathbb{R}^{n+1}$, then for any rescaling factor $\rho > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$, $\{\rho(\Sigma_{t_0+\frac{t}{\rho^2}} - x_0)\}_{t\in\mathbb{R}}$ is also a mean curvature flow.

Example 2.2.1. $M_t = \{|x|^2 = -2nt | x \in \mathbb{R}^{n+1}\}_{t\leq 0}$ *is the evolution of round hyperspheres centered*

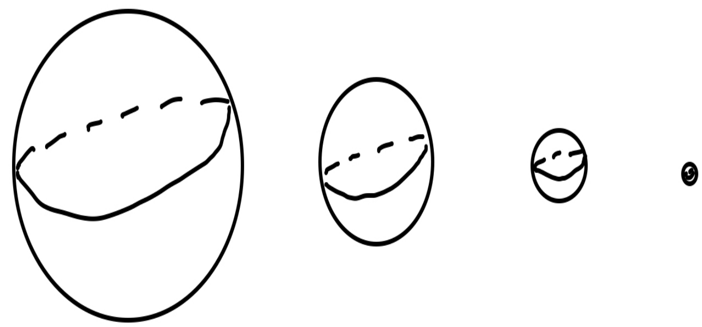

Figure 2.1: Evolution of round spheres

at the origin and that become extinct at the space time origin $(0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}$. *See figure 2.1 below.*

An important tool in studying the equation is the parabolic maximum principle. As a consequence of the maximum principle, we have the avoidance principle for mean curvature flow: If we start from two initially disjoint hypersurfaces, they will stay disjoint for all later times. We can see by the avoidance principle that singularities are unavoidable in mean curvature flow. In fact, for any smooth closed hypersurface, we can enclose it by a large enough sphere centered at the origin. By the Example 2.2.1, the sphere will shrink to a point in finite time. The avoidance principle tells us that the evolving hypersurface will stay enclosed by the round sphere all the time, and thus must also develop a singularity in finite time.

For the case $n = 1$, namely the curve shortening flow. The evolution of any close simple curve will stay embedded and become rounder and rounder until they shrinks to a round point, which is the first and only singularity of the flow, see [21, 23]. However, in higher dimensions, there are more non-trivial singularities in the mean curvature flow. For example, Grayson [22] showed that an initial surface with the shape of a dumbbell (2 large spherical end connected by a thin neck) will develop a neckpinch singularity before extinction. This was reproved by Angenent [1] using a avoidance principle argument with a shrinking torus.

An important tool in studying the singularities and regularity theory of mean curvature flow is the Huisken's montonicity formula.

In [24], Huisken discovered that

$$\frac{d}{dt}\int_{\Sigma_t} \rho(x,t) = -\int_{\Sigma_t} \rho|\mathbf{H}_{\Sigma_t} - \frac{1}{2t}\mathbf{x}^{\perp}|^2, \ (t<0) \tag{2.2.1}$$

where $\rho(x,t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}}$ is the backward heat kernel.

Using the monotonicity formula, Ilmanen [26] and White [38] showed that by parabolically scaling and zooming in a singularity of the flow and translting the singular point to space-time origin, the mean curvature flow converges weakly in the sense of measure to a limit flow. This limit flow is called a tangent flow at the singularity, a parabolic analog to the concept of tangent cone at the singularities in minimal surface theory. This tangent flow is a soliton flow that satisfies the equation

$$\mathbf{H}_{\Sigma_t} - \frac{1}{2t}\mathbf{x}^{\perp} = 0.$$

Such soliton solution evolves under mean curvature flow by self-similar shrinking

$$\Sigma_t = \sqrt{-t}\Sigma_{-1},\ (t < 0)$$
$$\Sigma_{-1} = \Sigma$$

It is not known however, whether the tangent flow at a singular point is unique, i.e. independent of rescaling sequence.

2.3 Self-shrinkers

The -1 time slice of a tangent flow to a singularity of mean curvature flow is a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ that satisfies the elliptic equation

$$\mathbf{H}_{\Sigma} + \frac{1}{2}\mathbf{x}^{\perp} = 0$$

Such hypersurfaces are called self-shrinkers. As we have seen by the parabolic zooming in procedure, self-shrinkers are infinitesimal models of singularities of mean curvature flow.

Although it's tempting to classify all these singularity models, it is impossible to get a complete classification of self-shrinkers at present since there are abundance of examples with complicated topology and geometry. See for example of the following picture of self-shrinkers (whose existence was conjectured by Ilmanen) with arbitrary high genus constructed by Kapoleas, Kleene and Moller [27] using glueing techinques.

From another point of view however, self-shrinkers can also be viewed as minimal hypersurfaces in a Gaussian metric that is conformal to the Euclidean metric. More precisely, self-shrinkers are minimal hypersurfaces in the metric $(\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{2n}}\delta_{ij})$ and the equation (2.3) is the Euler-Lagrange equation for the Gaussian weighted area functional $F(\Sigma) = \int_{\Sigma} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}$.

We can also computed the second variation formula for this functional (see e.g. [10]). For any

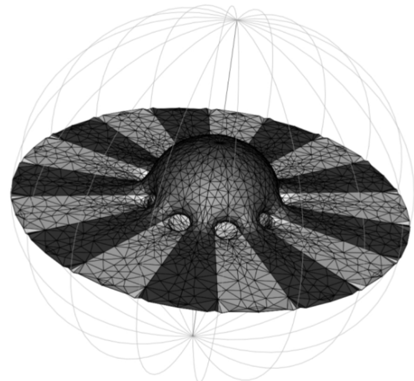

Figure 2.2: High genus self-shrinkers
Picture taken from [27]

compactly supported normal variation $\Sigma_t = \Sigma + f \cdot \mathbf{n}$ at a critical point Σ_0 (namely a self-shrinker Σ_0), the second variation is

$$\frac{d^2}{ds^2} F(\Sigma + s \cdot f\nu) = -\int_{\Sigma} fLf e^{-\frac{|x|^2}{4}} d\mu_{\Sigma}.$$

Here f is a compactly supported function and $\mathbf{n}$ is the unit normal vector field. And the Jacobi operator (or stability operator) is

$$L = \Delta_{\Sigma} + |A|^2 + \frac{1}{2} - \frac{1}{2} < x, \nabla_{\Sigma}(\cdot) >$$

where A is the second fundamental form of Σ.

One can use the test function $f \equiv 1$ to see that there are always some directions to decrease the weighted Gaussian area and self-shrinkers are all unstable in this sense.

Other important eigenfunctions of the stability operator include the scalar mean curvature H and the normal component $< \mathbf{v}, \mathbf{n} >$ of any constant vector field $\mathbf{v}$, where we have

$$\begin{aligned} LH &= H \\ L < \mathbf{v}, \mathbf{n} > &= \frac{1}{2} < \mathbf{v}, \mathbf{n} > \end{aligned} \tag{2.3.1}$$

Recall that the Jacobi operator for minimal hypersurfaces in general ambient manifold is

$$L = \Delta_{\Sigma} + |A|^2 + Ric(\mathbf{n}, \mathbf{n}).$$

Where Δ_Σ denotes the Laplacian on the minimal hypersurface Σ and $\mathbf{n}$ is a normal vector field on Σ. So if the Ricci curvature is positive, then the manifold admits no stable minimal surfaces by using the constant functions as test function. For example, there are no stable minimal hypersurfaces in the round spheres $\mathbb{S}^{n+1}$. In fact, it can be computed that Gaussian metric has positive weighted Ricci curvature in the sense of Bakry-Emery. And there are some similarity between the minimal surfaces in round spheres and self-shrinkers in $\mathbb{R}^{n+1}$. The third topic in this thesis is a result on self-shrinking torus in $\mathbb{R}^3$ that is analogous to the corresponding result about minimal tori in the round sphere $\mathbb{S}^3$.

2.4 The Colding-Minicozzi entropy

In [10], Colding-Minicozzi introduced the entropy functional $\lambda(\Sigma)$ for such a hypersurface when studying generic singularities of the mean curvature flow. It's a natural geometric quantity that measures the complexity of a hypersurface, and is defined by

$$\lambda(\Sigma) = \sup_{(\mathbf{y},\rho)\in\mathbb{R}^{n+1}\times\mathbb{R}} F(\rho\Sigma + \mathbf{y}) \tag{2.4.1}$$

where F is the Gaussian area of Σ defined by

$$F(\Sigma) = F_{0,1}(\Sigma) = \int_\Sigma \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}. \tag{2.4.2}$$

In general

$$F_{x_0,r}(\Sigma) = \int_\Sigma \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4r}}.$$

By definition the entropy is a scale and translation invariant quantity. It is also a lower semi-continuous functional on the space of hypersurfaces because it's taken supremum in the definition. The Gaussian weight is normalized so that the entropy of a flat hyperplane is equal to 1. It is easily seen that hyperplanes minimize entropy among all the immersed C^1 hypersurfaces by observing that every point has a well-defined tangent plane with entropy at least 1.

Self-shrinkers are critical points for the Gaussian area functional F. The entropy of a self-shrinker is equal to its Gaussian area according to computations in [10]. The associated flow for the self-shrinker has constant entropy independent of time. By Huisken's monotonicity formula, the entropy is non-increasing along a MCF.

As seen in the previous section, there are no stable self-shrinkers in the classical sense of minimal surface theory. Colding and Minicozzi defined two other notions of stability.

Definition 1. *A self-shrinker is said to be F-stable if for every one-parameter family of variations* Σ_s *of* Σ_0*, there exists variations* x_s *of 0 and* r_s *of 1 so that*

$$F'' = (F_{x_s,r_s}(\Sigma))'' \geq 0.$$

Definition 2. *A self-shrinker is said to be entropy-stable if it is a local minima of the entropy functional.*

From a dynamical point of view, generic singularities of mean curvature flow shall be entropy stable. Among other things, Colding and Minicozzi were able to show that the only F-stable self-shrinkers in all dimensions are round spheres and the only entropy-stable self-shrinkers in all dimensions are either round spheres all generalized round cylinders of the form $\sqrt{2n}\mathbb{S}^n, \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$, solving a long standing conjecture of Huisken.

3

Hausdorff stability of round spheres under small-entropy perturbation

In this chapter, we record my work on the Hausdorff stability of round spheres in [35]. We show that if the entropy of a hypersurface is close to that of the round sphere, it must be close to the round spheres in the Hausdorff sense up to rescaling and translation.

3.1 Main result

The first result we showed is the Hausdorff stability for the round n-sphere, generalizing the result of [2] to closed hypersurfaces in $\mathbb{R}^{n+1}$.

Theorem 1. *For any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that, if Σ is a closed hypersurface in $\mathbb{R}^{n+1}$ with entropy $\lambda(\Sigma) < \lambda(\mathbb{S}^n) + \delta$, then*

$$\inf_{\rho>0,\mathbf{y}\in\mathbb{R}^{n+1}} \operatorname{dist}_H(\frac{1}{\rho}\Sigma - \mathbf{y}, \mathbb{S}^n) < \epsilon,$$

or in other words, if a sequence of closed hypersurfaces with entropy converging to that of the round-spheres, then after recentering and rescaling, the sequence must converge to the round spheres in the Hausdorff distance.

It gives a quantitative version of the rigidity result for round hypser-spheres.

3.2 Weak mean curvature flows

In this section, we gather various notions of weak mean curvature flows and prove some properties of them that will be used in this note. We mostly follow the formulations in [25].

3.2.1 Brakke flows

An n-dimensional Brakke flow (Brakke motion) $\mathcal{K}$ in $\mathbb{R}^{n+1}$ is a family of Radon measures $\mathcal{K} = \{\mu_t\}_{t\in I}$, $\mu_t \in \mathbf{IM}_n(\mathbb{R}^{n+1})$, such that

(1) for a.e. t, $\mu_t = \mu(V_t)$ for some varifold $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ so that the first variation:

$$\delta V_t(X) = -\int \mathbf{H}(x)\cdot X d\mu_t$$

where $\mathbf{H}$ is the weak mean curvature vector field for a varifold.

(2) for any test function $f \in C^1_c(\mathbb{R}^{n+1}\times[a,b])$ and $f \geq 0$

$$\int f(\cdot,b)d\mu_b - \int f(\cdot,a)d\mu_a \leq \int_a^b \int (-|H|^2 f + \mathbf{H}\cdot\nabla f + \frac{\partial f}{\partial t})d\mu_t dt \tag{3.2.1}$$

A smooth mean curvature flow is automatically a Brakke motion with the inequality in (3.2.1) becoming an equality.

A Brakke flow $\{\mu_t\}_{t\in\mathbb{R}}$ is called eternal if $\mathrm{spt}\mu_t \neq \emptyset$ for all $t\in\mathbb{R}$.

We restrict our attention to n-dimensional Brakke flows $\{\mu_t\}_{t\in I}$ in $\mathbb{R}^{n+1}$ with bounded area ratios, i.e., for which there is a $C<\infty$ so that for all $t\in I$,

$$\sup_{x\in\mathbb{R}^{n+1}}\sup_{R>0}\frac{\mu_t(B_R(x))}{R^n} \leq C \tag{3.2.2}$$

Ilmanen ([26] Lemma 7) observed that the monotonicity formula of Huisken [24] could be extended to the class of Brakke flows with initial data that has bounded area ratio:

$$\int \rho(x,t_2)d\mu_2 - \int \rho(x,t_1)d\mu_1 \leq -\int_{t_1}^{t_2}\int \rho|\mathbf{H} - \frac{1}{2t}\mathbf{x}^\perp|^2 d\mu_t(x)dt \tag{3.2.3}$$

for $t_1 < t_2 < 0$.

As a corollary, bounded area ratio at an initial time will be of bounded area ratio with the same constant in later time.

Given a flow with bounded area ratio, we define the Huisken's density $\Theta_{(x_0,t_0)}(\{\mu_t\})$ at the space-time point (x_0, t_0) to be

$$\Theta_{(x_0,t_0)}(\{\mu_t\}) = \lim_{s\to t_0^-} \int \frac{1}{(-4\pi(s-t_0))^{\frac{n}{2}}} e^{\frac{|x-x_0|^2}{(s-t_0)}} d\mu_s(x)$$

which is upper semi-continuous by the monotonicity.

The entropy of a Brakke flow $\mathcal{K} = \{\mu_t\}_{t\in I}$ is defined by $\lambda(\mathcal{K}) = \sup_{t\in I} \lambda(\mu_t)$. It is a lower semi-continuous functional.

Remark 1. *It is not hard to see that, for a Radon measure, bounded area ratios are equivalent to finite entropy.*

Definition 3. *Let $\mathcal{K}_i = \{\mu_{i,t}\}_{t\geq t_0}$ be a sequence of integral Brakke flows, we say $\mathcal{K}_i$ converges to* $=\{\mu_t\}_{t\geq t_0}$*, if*

(1) $\mu_{i,t} \to \mu_t$ for all $t \geq t_0$

(2) for a.e. $t \geq t_0$, there is a subsequence $i(k)$, depending on t, so that $V_{\mu_{i(k),t}} \to V_{\mu_t}$

Convergence for flows with varying time intervals is defined analogously.

Brakke flows with uniform local mass bound have a good compactness theorem.

Theorem 2. *([25] section 7, cf. [7] chapter 4)*

Let $\mathcal{K}_i = \{\mu_{i,t}\}_{t\geq t_0}$ be a sequence of n-dimensional integral Brakke flows so that for all bounded open $U \subset \mathbb{R}^{n+1}$,

$$\sup_i \sup_{t\in[t_0,\infty)} \mu_{i,t}(U) \leq C(U) < \infty$$

.

There is a subsequence $i(k)$ and an integral Brakke flow $\mathcal{K}$ so that $\mathcal{K}_{i(k)} \to \mathcal{K}$.

In particular, the compactness theorem works for sequence of flows with a uniform entropy bound.

In [7], Brakke developed partial regularity theorem for Brakke flows. Later, White [39] simplified the proof for a special, but large class of Brakke flow, which include the class we use here. We will make use of a corollary of their theorem:

Proposition 1. *(Proposition 3.7 of [3]) Let $\{\mu_{i,t}\}_{t\geq t_0}$ be a sequence of integral Brakke flows converging to a limit integral Brakke flow $\{\mu_t\}_{t\geq t_0}$. If the limit flow is regular (smooth) in $B_R(y) \times (t_1, t_2)$, then*

(1) for each $t_1 < t < t_2$, $\mathrm{spt}(\mu_{i,t}) \to \mathrm{spt}(\mu_t)$ in $C^\infty_{loc}(B_R(y))$

(2) given $\epsilon > 0$, there is an $i_0 = i_0(\epsilon, \{\mu_t\})$ so that if $i > i_0$, $\mu_{i,t}$ is regular (smooth) in $B_{R-\epsilon}(y) \times (t_1 + \epsilon, t_2)$

Denote the parabolic rescaling and translation of a Brakke flow $\mathcal{K} = \{\mu_t\}$ by

$$D_\rho \mathcal{K} = \{\rho \mu_{\frac{t}{\rho^2}}\}$$

$$\mathcal{K} - (x_0, t_0) = \{\mu_{t+t_0} - x_0\}$$

Using Huisken's monotonicity formula, Ilmanen ([26] Lemma 8) proved that: if $\Theta_{(x_0,t_0)} > 0$ (this is equivalent to $\Theta_{(x_0,t_0)} \geq 1$), then there is a subsequence $\rho_i \to \infty$ such that $D_{\rho_i}(\mathcal{K} - (x_0, t_0)) \to \tilde{\mathcal{K}}$. Such a limit flow $\tilde{\mathcal{K}}$ is called a tangent flow at (x_0, t_0), and it is a backward self-shrinker for negative time.

3.2.2 Enhanced motions and matching motions

For $T \in \mathbf{I}_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R})$, denote $T \lfloor (\mathbb{R}^{n+1} \times [a, b]) = T_{a \leq t \leq b}$, $\partial T_{a \leq t \leq b} = T_a - T_b$, and $\partial T_{t \geq a} = T_a$.

A pair $(T, \mathcal{K})$ is called an enhanced motion, if $T \in \mathbf{I}_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R})$ and $\mathcal{K} = \{\mu_t\}_{t \in \mathbb{R}}$ satisfying

(1) $\partial T = 0$ and $\partial(T_{t \geq s}) = T_s$ and $T_t \in \mathbf{I}_n(\mathbb{R}^{n+1})$ for each time slice t

(2) $\partial T_t = 0$ for all t and $t \mapsto T_t$ is continuous in the flat topology

(3) $\mathcal{K} = \{\mu_t\}_{t \in \mathbb{R}}$ is a Brakke motion

(4) $\mu_{T_t} \leq \mu_t$ for all t and they are compatible for a.e. t

T is the undercurrent and $\mathcal{K}$ is the overflow.

An enhanced motion $(T, \mathcal{K})_{t \geq 0}$ with initial data T_0 is one that condition (1) above replaced by

(1') $\partial T = T_0, \mu_{T_0} = \mu_0$, and $\partial(T_{t > s}) = T_s$ and $T_t \in \mathbf{I}_n(\mathbb{R}^{n+1})$ for each time slice t.

An enhanced motion in a space-time open subset $U \times I \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is defined by replacing the space-time domains $\mathbb{R}^{n+1}$ and $\mathbb{R}$ in the 4 items by U and I repectively.

The enhanced motion $(T, \mathcal{K})$ is called a matching motion if $\mu_{T_t} = \mu_t = \mu_{V_t}$ for a.e. t. So for matching motions, we do not distinguish $\mu_{T_t}, \mu_t, \mu_{V_t}$ for a.e. t. A smooth flow automatically gives rise to a matching motion.

The existence of an enhanced motion with initial data a cycle was proved by Ilmanen in [25] using an elliptic regularization procedure, and reproved by White in [40]. The continuity in flat topology (2) was not explicitly stated in [25], but was pointed out in [40].

There are corresponding compactness theorems for integral currents and Brakke flows with finite mass, but we cannot guarantee that the limit of matching motions is still a matching motion in general due to lower semi-continuity of the map V_2. A counter example is the blow-down limit of a Grim-Reaper translating soliton of (smooth) mean curvature flow is a quasi-static multiplicity 2 plane with zero undercurrent, i.e. it is not matching.

However, we can rule this out for small entropy and get a compactness theorem for matching motions with low entropy.

Theorem 3. *Let $(T_i, \mathcal{K}_i)$ be a sequence of matching motions in $\mathbb{R}^{n+1} \times I$, that converge to an enhanced motion $(T, \mathcal{K})$ in $\mathbb{R}^{n+1} \times I$. If $\lambda(\mathcal{K}) < 2$, then the limit is also a matching motion.*

To prove the theorem above, we need a lemma about compatibility of integral currents and varifolds by White [40].

Lemma 1. *(Theorem 3.6 of [40]) Suppose V_i is a sequence of integer multiplicity rectifiable varifolds that converge with locally bounded first variation to an integer multiplicity rectifiable varifold, V. And T_i is a sequence of integral currents such that V_i and T_i are compatible. If the boundaries, ∂T_i, converge (in the integral flat topology) to a limit integral flat chain, then there is a subsequence $i(k)$ such that $T_{i(k)}$ converge to an integral current T. Furthermore V and T must then be compatible.*

Proof. (of Theorem 3)

We have $\mathcal{K}_i = \{\mu_{i,t}\} \to \{\mu_t\} = \mathcal{K}$ as Brakke flows and $T_i \to T$ as currents.

By Brakke's convergence, there is a set S_1 with $\mathcal{L}^1(S_1) = 0$ (where $\mathcal{L}^1$ denote the Lebesgue measure), for all $t \in I \setminus S_1$, there is a subsequence $i(k)_t$, depending on t, such that, $V_{i(k),t} \to V_{\mu_t}$ with locally bounded first variation (Lemma 4.3 of [40]).

By a slicing lemma of White (pp. 208 of [37]), there is a another set S_2 with $\mathcal{L}^1(S_2) = 0$, for all $t \in I \setminus (S_1 \cup S_2)$, there is a further subsequence $i(k(j))$, also depending on t, such that $T_{i(k(j)),t} \to T_t$.

Moreover, because $(T_i, \mathcal{K}_i)$ are matching motions, there is a set S_3 with $\mathcal{L}^1(S_3) = 0$, for all $t \in I \setminus (S_1 \cup S_2 \cup S_3)$, $\mu_{T_{i(k(j)),t}} = \mu_{i(k(j)),t}$ and $V_{T_{i(k(j)),t}} = V_{i(k(j)),t}$. Namely, $T_{i(k(j)),t}$ and $V_{i(k(j)),t}$ are compatible.

By definition of matching motions we have $\partial T_{i(k(j)),t} = 0$ for all t, so the condition of Lemma 1 is satisfied. Thus, for each $t \in I \setminus (S_1 \cup S_2 \cup S_3)$, we can extract a further subsequence $i(k(j(l)))$ such that, $\lim_{i(k(j(l)))\to\infty} T_{i(k(j(l))),t}$ is compatible with $\lim_{i(k(j))\to\infty} V_{i(k(j)),t} = V_{\mu_t}$.

Since the limit of a subsequence must be same as the limit of original sequence, we have $\lim_{i(k(j(l)))\to\infty} T_{i(k(j(l))),t} = \lim_{i(k(j))\to\infty} T_{i(k(j)),t} = T_t$.

And T_t is compatible with V_{μ_t}, namely

$$V_{\mu_t} = V_{T_t} + 2W_t$$

for $t \in I \setminus (S_1 \cup S_2 \cup S_3)$ and some $W_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$.

Claim 3.2.1. *For any $W \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $W \neq 0$, we have $\lambda(W) \geq 1$*

Proof. (of Claim 3.2.1)

Because W is rectifiable, it is a.e. a C^1 submanifold with integer multiplicity. For such a point x_0 with C^1 submanifold structure, it has a tangent plane. And $\lim_{\rho\to\infty} F(\rho(\Sigma - x_0))$ will become the Euclidean density at this point, which is at least 1. □

Now we have for $t \in I \setminus (S_1 \cup S_2 \cup S_3)$

$$2\lambda(W_t) \le \lambda(V_{T_t} + 2W_t) = \lambda(V_{\mu_t}) \le \lambda(\mathcal{K}) < 2$$

which forces $W_t = 0$ by the Claim 3.2.1 above.

So we have for a.e. t

$$V_{\mu_t} = V_{T_t}$$

The limit is also a matching motion.

□

3.2.3 Level-set flow and canonical boundary motions

In order to get a matching motion from a generic surface, we will need another notion of set theoretic weak flow called the level-set flow. The mathematical theory of level-set flow was developed by Chen-Giga-Goto [9] and Evans-Spruck [14, 15, 16, 17]. We follow the formulation of level-set flow of Evans-Spruck [14].

Let Γ be a compact non-empty subset of $\mathbb{R}^{n+1}$. Select a continuous function μ_0 so that $\Gamma = \{x : u_0(x) = 0\}$ and there are constants $C, R > 0$ so that

$$u_0 = -C \quad on\ \{x \in \mathbb{R}^{n+1} : |x| \ge R\} \tag{3.2.4}$$

for some sufficiently large R. In particular, $\{u_0 \ge a > -C\}$ is compact. In [14], Evans-Spruck established the existence and uniqueness of viscosity weak solutions to the initial value problem:

$$\begin{cases} u_t = \Sigma_{i,j=1}^{n+1}(\delta_{ij} - u_{x_i}u_{x_j}|Du|^{-2})u_{x_i x_j} \quad on\ \mathbb{R}^{n+1} \times (0,\infty) \\ u = u_0 \quad on\ \mathbb{R}^{n+1} \times \{0\} \end{cases} \tag{3.2.5}$$

Setting $\Gamma_t = \{x : u(x,t) = 0\}$, define $\{\Gamma_t\}_{t\ge 0}$ to be the level-set flow of Γ. It is justified in [14] that the $\{\Gamma_t\}$ is independent of the choice of u_0.

Level-set flow has a uniqueness property and an avoidance principle. But it may fatten up in later time, namely a level-set flow in $\mathbb{R}^{n+1}$ may develop some time slices that have non-zero (n+1)-dimensional Hausdorff measure. But we have the following genericity of non-fattening.

Proposition 2. *(11.3 of [25]) For any closed hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$, and any $\epsilon > 0$, and any given $k > 0$, there is a small perturbation Σ' of Σ, which is a graph u over Σ with $||u||_{C^k} < \epsilon$ and such that the level-set flow starting from Σ' is non-fattening.*

A non-fattening level-set flow gives rise to a matching motion. (see [25] pp. 55)

We will make use of the existence of a special kind of matching motion called the canonical boundary motion from these generic surfaces (see [25], section 11).

Using definitions from [33], $\partial^* E$ is the reduced boundary of E. If E is of locally finite perimeter, then $\mathcal{H}^n \lfloor \partial^* E \in \mathbf{IM}_n(\mathbb{R}^{n+1})$.

Definition 4. *A $\mu \in \mathbf{IM}_n(\mathbb{R}^{n+1})$ is a compact boundary measure, if there is a bounded open non-empty subset $E \subset \mathbb{R}^{n+1}$ of locally finite perimeter so that $\mathrm{spt}(\mu) = \partial E$ and $\mu = \mathcal{H}^n \lfloor \partial^* E$. Such a set E is called the interior of μ.*

Ilmanen synthesized both notions of weak flows and show that there is a canonical way to associate a Brakke flow to a level-set flow for a large class of initial sets. ([25], section 11)

Definition 5. *Given a compact boundary measure μ_0 with interior E_0, a canonical boundary motion of μ_0 is a pair $(E, \mathcal{K})$ consisting of an open bounded subset E of $\mathbb{R}^{n+1} \times \mathbb{R}^+$ of finite perimeter and a Brakke flow $\mathcal{K} = \{\mu_t\}_{t \geq t_0}$ so that:*

(1) $E = \{(x,t) : u(x,t) > 0\}$, where u solves equation (3.2.5) with $E_0 = \{x : u_0(x) > 0\}$ and $\partial E_0 = \{x : u_0(x) = 0\}$

(2) each $E_t = \{x : (x,t) \in E\}$ is of finite perimeter and $\mu_t = \mathcal{H}^n \lfloor \partial^* E_t$.

In particular, a canonical boundary motion is a non-fattening level-set flow. Ilmanen proved the existence of canonical boundary motions (Theorem 11.4 of [25]). We need a weaker version of it

Theorem 4. *(Theorem 11.4 of [25]) If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a closed hypersurface such that the level-set flow is non-fattening, then there is a canonical boundary motion starting from Σ. In particular, it is a matching motion.*

The following uniqueness theorem of the flow of round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ will be used in the proof of the main theorem (when applying Theorem 1, we need the limit flow to be regular). It was not explicitly stated in [3], but can be drawn as a corollary of what was proved in that paper.

Theorem 5. *If $(T, \mathcal{K})$ is a matching motion in $\mathbb{R}^{n+1} \times [-1, \infty)$, $\mathcal{K} = \{\mu_t\}_{t\in[-1,\infty)}$, $\lambda(\mathcal{K}) = \Lambda_n$. Suppose it is the limit of a sequence of compact canonical boundary motions: $\mathcal{K} = \lim \mathcal{K}_i$, with each $\mathcal{K}_i$ becoming extinct at $(0,0) \in \mathbb{R}^{n+1} \times [-1, \infty)$, $\lambda(\mathcal{K}_i) \to \Lambda_n$, then $(T, \mathcal{K})$ is the regular flow of a round n-sphere.*

Proof. By Lemma 5.1 of [3], the extinction time of $\mathcal{K}_i$ are collapsed (Definition 4.9 of [3]). By Theorem 1.3 of the same paper [3], when i is large enough, the only possible tangent flow at the extinction time are round spheres. Proposition 4.10 of [3] implies that being collapsed is a closed condition, so the extinction time of $\mathcal{K}$ is also collapsed, which can only be a round sphere by the entropy bound.

Since the entropy is monotonic non-increasing under the flow, we conclude that it is constant over time and equal to that of a round sphere and thus a self-shrinker by monotonicity. Combining with the fact that it's extinct at a round-sphere tangent flow at $(0,0)$ in space-time, it must be the flow of a round shrinking n-sphere.

□

3.3 Properties of low entropy flows

Some of the results in this section can be made stronger by weakening the entropy bounds in the conditions, but the versions here are enough for our purpose.

According to Proposition 4.3 of [3], any $\mu \in \mathcal{CSM}_n(\frac{3}{2})$ is a compact boundary measure. In particular, if $n \geq 3$, any $\mu \in \mathcal{CSM}_n(\Lambda_{n-1})$ is a compact boundary measure. As the corresponding results in dimension 2 is already known, we restrict ourselves to $n \geq 3$ in this section.

For $\mu \in \mathcal{SM}_n(\mathbb{R}^{n+1})$, we call $\mathcal{K} = \{\mu_t\}_{t\in\mathbb{R}}$ an associated Brakke flow to μ if $\mu_t = \sqrt{-t}\mu$ for $t < 0$. An associated matching motion to a self-shrinking measure is one whose associated overflow is an associated Brakke flow. By Theorem 3, any tangent flow of a matching motion with entropy bounded by 2 is an associated matching motion to a self-shrinking measure.

Lemma 2. *(Lemma 4.4 of [3]) For $\mu \in \mathcal{SM}_n(\mathbb{R}^{n+1})$ with $\lambda(\mu) < \infty$, let $\mathcal{K}$ be an associated Brakke flow to μ. If there is a $y \in \mathbb{R}^{n+1} \setminus \{0\}$ with $\Theta_{(y,0)} \geq 1$ and $\mathcal{T}$ is a tangent flow of $\mathcal{K}$ at $(y, 0)$, then $\mathcal{T}$ splits off a line backward in time, that is $\mathcal{T}_{t\leq 0} = \{\tilde{\mu}_t\}_{t\leq 0} = \{\nu_t \times \mathbb{R}\}_{t\leq 0}$, for some $\nu_t \in \mathbf{IM}_{n-1}(\mathbb{R}^n)$ and $\nu_{-1} \in \mathcal{S}M_{n-1}(\mathbb{R}^n)$.*

Proof. For the $y \neq 0$ with $\Theta_{(y,0)} \geq 1$, and $\mathcal{T}$ being a tangent flow at $(y, 0)$, there exists a sequence $\rho_i \to \infty$ such that $\rho_i(\mathcal{K} - (y, 0)) \to \mathcal{T}$. By the self-similarity of $\mathcal{K}$, we have, for any $\tau \in \mathbb{R}$

$$
\begin{aligned}
&\mathcal{T}_{t\le 0} - (\tau y, 0)\\
=&\lim_{i\to\infty} D_{\rho_i}(\mathcal{K}_{t\le 0} - (y + \frac{1}{\rho_i}\tau y, 0))\\
=&\lim_{i\to\infty} D_{\rho_i(1+\frac{1}{\rho_i}\tau)}[D_{(1+\frac{1}{\rho_i}\tau)^{-1}}\mathcal{K}_{t\le 0} - (y, 0)]\\
=&\lim_{i\to\infty} D_{\rho_i(1+\frac{1}{\rho_i}\tau)}(\mathcal{K}_{t\le 0} - (y, 0))\\
=&\lim_{i\to\infty} D_{\rho_i}(\mathcal{K}_{t\le 0} - (y, 0))\\
=&\mathcal{T}_{t\le 0}
\end{aligned}
$$

where we used the fact that $\lim_{i\to\infty}(1 + \frac{1}{\rho_i}\tau) = 1$ and the backward self-similarity of $\mathcal{K}$.

Since τ is arbitrary, we conclude that the tangent flow splits off a line in the direction of y backward in time. □

Lemma 3. *If $(\tilde{T}, \tilde{\mathcal{K}})$ is an associated matching motion of an asymptotic conical self-shrinker Σ^3 that is an tangent flow of a matching motion with entropy less than 2, then it cannot be extinct at time 0.*

Proof. Because Σ is asymptotic to a regular cone, there is $(\tilde{x}_0, 0)$, $\tilde{x}_0 \in \mathbb{R}^{n+1} \setminus \{0\}$ in the regular support of $\mu_{\tilde{T}_0}$, $(\Theta_{(\tilde{x}_0,0)} = 1)$. Namely, a tangent flow at $(\tilde{x}_0, 0)$ is a multiplicity 1 plane for negative time. If 0 is the extinction time, then the tangent flow must also be 0 for positive time. We get a quasi-static multiplicity 1 plane as a tangent flow, which is not a matching motion, a contradiction to Theorem 3.

□

Proposition 3. *For each n, there exists a $\delta(n)$ such that: If $(T, \mathcal{K})$ is a matching motion in $\mathbb{R}^{n+1}$ with $\lambda(\mathcal{K}) \le \Lambda_n + \delta(n)$ that becomes extinct at time t_0 and $\Theta_{(x_0,t_0)} \ge 1$ for some $x_0 \in \mathbb{R}^{n+1}$, then any tangent flow at (x_0, t_0) is the round n-sphere.*

Proof. Since $n \ge 3$, if we choose $\delta(n) < \Lambda_{n-1} - \Lambda_n$, any element in $\mathcal{CSM}_n(\Lambda_n + \delta(n))$ is a compact boundary measure. By the results Corollary 6.5 of [3] for dimensions $2 \le n \le 6$ and Corollary 2.9 of [42] for all higher dimensions, we can choose some $\delta(n) < (\Lambda_{n-1} - \Lambda_n)$ so that the only element in $\mathcal{CSM}_n(\Lambda_n + \delta(n))$ that is a compact boundary measure is the round sphere.

For $n = 3$, by Proposition 3.3 of [4], if $\mu \in \mathcal{SM}_3(\Lambda)$ does not have compact support, then $\mu = \mu_{\Sigma^3}$ where Σ^3 is a regular self-shrinker that is asymptotic to a regular cone (the link of the asymptotic cone is a smooth embedded hypersurface in $\mathbb{S}^3$).

So for $n = 3$, if t_0 is the extinction time and $\Theta_{(x_0,t_0)} > 0$, then a tangent flow at (x_0, t_0) is a matching motion by Theorem 3, and extinct at time 0 because $(T, \mathcal{K})$ is extinct at t_0. Combining Lemma 3, we conclude that a tangent flow at (x_0, t_0) is round 3-sphere.

For dimension $n \geq 4$, since we don't have these regularity result, we argue by induction. Suppose we know that for $k = 3, ..., n-1$, any k-dimensional self-shrinking matching motion that is not a sphere cannot be extinct at time 0.

If an extinction-time tangent flow of $(T, \mathcal{K})$, is $\mu^n = \lim_{i\to\infty} D_{\rho_i}(\mathcal{K} - (x_0, t_0))$ for some $\rho_i \to \infty$, $\mu^n \in \mathcal{SM}_n(\Lambda_n + \delta(n))$, non-compact, with associated matching motion being $(\tilde{T}^n, \tilde{\mathcal{K}}^n)$, and that it's extinct at 0, we can choose $y_0^n \in \mathbb{R}^{n+1} - \{0\}$ such that $\Theta_{(y_0,0)}(\{\mu_{\tilde{T}_t^n}\}) > 0$, then any tangent flow at $(y_0, 0)$ splits off a line backward in time by Lemma 2, say it is $\{\nu_t \times \mathbb{R}\}$ for $t \leq 0$ and $\nu_{-1} \in \mathcal{SM}_{n-1}(\Lambda_n + \delta(n)) \subset \mathcal{SM}_{n-1}(\Lambda_{n-1})$. Since it's a tangent flow at the extinction time, $\{\nu_t\}$ must also become extinct at time 0, and is not the (n-1)-sphere by the entropy bound, contradicting the induction hypothesis, and thus we proved the Proposition. $\square$

We have the following straightforward consequence.

Corollary 1. *For the same $\delta(n)$, if $\mu \in \mathcal{SM}_n(\Lambda_n + \delta(n))$ has a non-compact support, and it has associated matching motion, then this matching motion cannot be extinct at time 0.*

Lemma 4. *Let $(T_i, \mathcal{K}_i = \{\mu_{i,t}\})$ be a sequence of matching motions in $\mathbb{R}^{n+1}$ converging to $(T, \mathcal{K} = \{\mu_t\})$, $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$ and*

$$0 \in \operatorname{spt}(\mu_{i,0})$$

for all i. If it does not develop a spherical singularity for $t \in (-R, R)$, $(R > 0)$, then

$$0 \in \operatorname{spt}(\mu_0)$$

Remark 2. *Without the condition that $(T, \mathcal{K})$ does not develop a spherical singularity for $t \in (-R, R)$, the lemma is false. For example we can choose a sequence of regular space-time points on the shrinking sphere that converges to its extinction space-time point.*

Proof. By upper semi-continuity of the Huisken's density, we have $\Theta_{(0,0)}(\mu_t) \geq 1$. If $0 \notin \operatorname{spt}(\mu_0)$, then there is a neighborhood $U \subset \mathbb{R}^{n+1}$ of 0 such that $U \cap \operatorname{spt}(\mu_0) = \emptyset$, and 0 is an extinction singularity for the flow $(T, \mathcal{K})$ restricted to U.

By Proposition 3, the tangent flow of $(T, \mathcal{K})$ at $(0, 0)$ is multiplicity-1 round sphere. By Brakke's regularity Proposition 1, for large enough i, $(T_i, \mathcal{K}_i)$ must also be a flow of a topological sphere that develops a spherical singularity before time $\frac{R}{2}$, a contradiction. $\square$

Proposition 4. *An ancient matching motion $(T, \mathcal{K})$ in $\mathbb{R}^{n+1}$ with $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$, where $\delta(n)$ is given in Lemma 5, is either eternal or the flow of a topological sphere.*

Proof. Suppose it's not eternal, it has extinction time t_0.

Choose (x_0, t_0) such that $\Theta_{(x_0,t_0)}(\mathcal{K}) \geq 1$. By the entropy bound and Brakke's compactness theorem, there is a blow-down sequence of flows $D_{\rho_i}(\mathcal{K} - (x_0, t_0))$, for some $\rho_i \to 0$, converging to a limit flow $\tilde{\mathcal{K}} = \{\tilde{\mu}_t\}$. The limit is a matching motion $(\tilde{T}, \tilde{\mathcal{K}})$ by Theorem 3 and extinct at $t = 0$. Huisken's monotonicity formula [24] implies that this limit flow is backwardly self-similar for $t < 0$.

By Corollary 1, $(\tilde{T}, \tilde{\mathcal{K}})$ must be the self-shrinking round sphere, $\tilde{\mu}_{-1} = \sqrt{2n}\mathbb{S}^n$. The convergence is multiplicity 1 by the entropy bound. So by Brakke's regularity Proposition 3, for large enough i, $D_{\rho_i}(\mathcal{K} - (x_0, t_0))$ is also the flow of a topological sphere. $\rho_i \mu_{-\frac{1}{\rho_i^2}} \to \sqrt{2n}\mathbb{S}^n$ in C^∞ as $\rho_i \to 0$.

$\square$

Remark 3. *In Proposition 3.2 of [2], they got a stronger classification in $\mathbb{R}^3$ by making use of the entropy lower bound for 2-dimensional asymptotic conical self-shrinker in [5]. That depend on a classification of genus 0 self-shrinkers by Brendle [8], the argument of which only works in dimension 2.*

Proposition 5. *The $\delta(n)$ can be chosen small enough so that: if $(T, \mathcal{K} = \{\mu_t\}_{t\in\mathbb{R}})$ is a matching motion in $\mathbb{R}^{n+1} \times \mathbb{R}$ with entropy $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$, that it develops a spherical singularity at (x_0, t_0), then the flow is extinct at time t_0 at the point x_0.*

Proof. Suppose not, there is a sequence of matching motions $(T_i, \mathcal{K}_i = \{\mu_{i,t}\})$, with entropy $\lambda(\mathcal{K}_i) < \Lambda_n + \frac{1}{i}$, that develops a spherical singularity at (x_i, t_i) but not extinct at time t_i. Without loss of generality, we can suppose $(x_i, t_i) = (0, 0)$, otherwise we can do a space-time translation to make this happen.

Since the flows are not extinct at $(0, 0)$, there is a point $(y_i, 0)$ such that $y_i \neq 0$ and $y_i \in \mathrm{spt}(\mu_{i,0})$.

We consider the rescaled flows $\tilde{\mathcal{K}}_i = D_{\frac{1}{|y_i|}}\mathcal{K}_i$. The new flows satisfy that $\Theta_{(0,0)}\tilde{\mathcal{K}}_i = \Lambda_n, \Theta_{(\frac{y_i}{|y_i|},0)} \geq 1$. By Brakke's compactness theorem, we can extract a subsequence $i(k)$ so that $\tilde{\mathcal{K}}_{i(k)} \to \mathcal{K}_\infty$, and $\frac{y_i}{|y_i|} \to u$. The limit flow $\mathcal{K}_\infty$ is also a matching motion by Theorem 3. Moreover, by the upper semi-continuity of Huisken's density and the lower semi-continuity of entropy, we have

$$\lambda(\mathcal{K}_\infty) = \Lambda_n$$
$$\Theta_{(0,0)} \geq \Lambda_n$$
$$\Theta_{(u,0)} \geq 1 \text{ for some u with } |u| = 1$$

But this is a contradiction, because by Huisken's monotonicity formula, for some time $t < 0$, the time t slice of the flow $\mathcal{K}_\infty$ has entropy strictly greater than Λ_n.

□

3.4 A uniform continuity estimate of Hausdorff distance

In this section, we will prove the following 2-sided clearing out lemma. It is a generalization of the classical Brakke's backward clearing out lemma, and under under an entropy bound we can show that it holds both backward and forward in time. The statement in the smooth sense is as follows:

Theorem 6. *There exists $\delta(n) > 0, C(n) > 0, \eta(n) > 0, \gamma(n) \in (0,1)$ so that: if $\{M_t^n\}$ is a a mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$ that reaches the space-time point $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$, with entropy $\lambda(M_t) \leq \lambda(\mathbb{S}^n) + \delta(n)$, and $M_t \neq \emptyset$ for all $t \in (t_0 - R^2, t_0 + R^2)$, then for all $0 < C\rho < \gamma R$*

$$\mathcal{H}^n(B_\rho(x_0) \cap M_{t_0 - C^2\rho^2}) \geq \eta\rho^n$$

and

$$\mathcal{H}^n(B_\rho(x_0) \cap M_{t_0 + C^2\rho^2}) \geq \eta\rho^n$$

where $\mathcal{H}^n$ denotes the n-dimensional volume on hypersurfaces.

We will still restrict ourselves to dimension $n \geq 3$ for convenience, the 2-dimensional case was already known in [2]. The following is Clearing out Lemma of Brakke ([7] Lemma 6.3, cf. 12.2 of [25], Proposition 4.23 of [13]). It can be formulated as follows

Lemma 5. *(12.2 of [25]) There are constants $\eta > 0, c_1 > 0$, depending on n such that, for any n-dimensional integral Brakke flow $\{\mu_t\}_{t\geq 0}, R > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1} \times [0, \infty)$, if*

$$\mu_{t_0}(B_R(x_0)) \leq \eta R^n,$$

then

$$\mu_{t_0+c_1R^2}(B_{\frac{R}{2}}(x_0)) = 0.$$

Proposition 6. *For $\delta(n)$ chosen satisfying Proposition 3 and Proposition 5, there is $C(n) > 0, \eta(n) > 0$ so that: if $(T, \mathcal{K} = \{\mu_t\})$ is an eternal matching motion in $\mathbb{R}^{n+1}$, with $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$ and $0 \in \operatorname{spt}(\mu_0)$. Then for all $\rho > 0$*

$$\mu_{-\rho^2}(B_{C\rho}(0)) \geq \eta(C\rho)^n \tag{3.4.1}$$

and

$$\mu_{\rho^2}(B_{C\rho}(0)) \geq \eta(C\rho)^n \tag{3.4.2}$$

Proof. We choose $C \geq \frac{1}{\sqrt{c_1}}$ according to Lemma 5. For any $\rho > 0$, let $R = \frac{\rho}{\sqrt{c_1}}, t_0 = -\rho^2, x_0 = 0$ and η smaller than one given in Lemma 5. We have

$$\mu_{-\rho^2}(B_{C\rho}(0)) \geq \eta(C\rho)^n$$

for otherwise by Lemma 5 we have $\mu_{t_0+c_1R^2}(B_{\frac{R}{2}}(x_0)) = \mu_0(B_{\frac{\rho}{2\sqrt{c_1}}}(0)) = 0$, contradicting the condition $0 \in \operatorname{spt}(\mu_0)$.

To prove (3.4.2), we use standard blow-up argument. Suppose not, there exists a sequence $C_i \to \infty, \eta_i \to 0$, satisfying $\eta_i C_i^n \to 0$, a sequence of eternal matching motions $(T_i, \mathcal{K}_i = \{\mu_{i,t}\})$ with $\lambda(\mathcal{K}_i) \leq \Lambda_n + \delta(n), 0 \in \operatorname{spt}(\mu_{i,0})$, and a sequence $\rho_i > 0$, such that

$$\mu_{\rho_i^2}(B_{C_i\rho_i}(0)) < \eta_i(C_i\rho_i)^n$$

We rescale the flows parabolically by factors $\frac{1}{\rho_i}$ to get $(\tilde{T}_i, \tilde{\mathcal{K}}_i = \{\tilde{\mu}_t\})$, where $\tilde{\mathcal{K}}_i = D_{\frac{1}{\rho_i}}\mathcal{K}_i$, which are also eternal flows as well. And they satisfy

$$\tilde{\mu}_1(B_{C_i}(0)) < \eta_i(C_i)^n \to 0$$

By Brakke's compactness theorem, there is a subsequence $i(k)$ such that $\tilde{\mathcal{K}}_{i(k)} \to \mathcal{K}_\infty$, which is also a matching motion $(T_\infty, \mathcal{K}_\infty)$ by Theorem 3.

Since $C_{i(k)} \to \infty$ and $\tilde{\mu}_1(B_{C_{i(k)}}(0)) < \eta_i(C_{i(k)})^n \to 0$, the limit flow $\mathcal{K}_\infty$ must be extinct before time $t = 1$, thus not eternal. By Theorem 4, the limit flow is the flow of a topological sphere.

The entropy bound gives the multiplicity of convergence is 1. By Brakke's regularity Proposition

1, the convergence is smooth and as graphs over topological spheres. For large enough $i(k)$, $\tilde{\mathcal{K}}_{i(k)}$ also develops a spherical singularity in finite time t. By Proposition 5, it must be extinct at the time a spherical singularity occur, contradicting the fact that they are eternal, and hence we proved the proposition.

□

Theorem 7. *For the same $\delta(n), C(n), \eta(n)$ chosen from Proposition 6, there is a $\gamma \in (0,1)$ so that: if $(T, \mathcal{K} = \{\mu_t\})$ is a matching motion in $\mathbb{R}^{n+1}$ with $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$, $\{\mu_t\}$ does not develop spherical singularities for $t \in (-R^2, R^2)$, and $0 \in \operatorname{spt}(\mu_0)$, then for all $\rho < \gamma R$*

$$\mu_{-\rho^2}(B_{C\rho}(0)) \geq \eta(C\rho)^n \tag{3.4.3}$$

and

$$\mu_{\rho^2}(B_{C\rho}(0)) \geq \eta(C\rho)^n \tag{3.4.4}$$

Remark 4. *Theorem 6 is a special case of Theorem 7 for smooth flows.*

Remark 5. *This 2-sided clearing out lemma is not true for flows $\{M_t\}$ with entropy $\lambda(\{M_t\}) \geq \lambda(\mathbb{S}^{n-1} \times \mathbb{R})$. Consider the rotational-symmetric translating "bowl" soliton, whose entropy is equal to $\lambda(\mathbb{S}^{n-1} \times \mathbb{R})$ and is rescaled so that the speed of translation is $\frac{1}{C^2}$. For any $\gamma \in (0,1)$, by choosing $\rho > 1, R > \frac{C\rho}{\gamma}$, and (x_0, t_0) to be the tip of the translating "bowl" soliton, we get a counter example. However, we speculate that the theorem should still hold under the entropy bound $\lambda(\{M_t\}) < \lambda(\mathbb{S}^{n-1} \times \mathbb{R})$.*

In the second topic of this thesis, we will record the joint work with Bernstein on the proof this conjecture in dimension 3 and 4.

Proof. Suppose not, then there is a sequence $\gamma_i \to 0$, a sequence of $R_i > 0$, a sequence of $\rho_i < \gamma_i R_i$, a sequence of matching motions $(T_i, \mathcal{K}_i = \{\mu_{i,t}\})$, with $\lambda(\mathcal{K}_i) \leq \Lambda_n + \delta(n)$, $\{\mu_{i,t}\}$ does not develop spherical singularities for $t \in (-R_i^2, R_i^2)$, $0 \in \operatorname{spt}(\mu_{i,0})$, such that

$$\mu_{i,-\rho_i^2}(B_{C\rho_i}) < \eta(C\rho_i)^n$$

or

$$\mu_{i,\rho_i^2}(B_{C\rho_i}) < \eta(C\rho_i)^n$$

We rescale the flows parabolically by factors $\frac{1}{\rho_i}$ and get $(\tilde{T}_i, \tilde{\mathcal{K}}_i)$, where $\tilde{\mathcal{K}}_i = D_{\frac{1}{\rho_i}} \mathcal{K}_i$, with

$\lambda(\tilde{\mathcal{K}}_i) \leq \Lambda_n + \delta(n)$, satisfying

$$\mu_{i,-1}(B_C) < \eta(C)^n$$

or

$$\mu_{i,1}(B_C) < \eta(C)^n$$

Because of rescaling, we also have $\{\tilde{\mu}_{i,t}\}$ does not develop spherical singularities for $t \in (-\frac{R_i^2}{\rho_i^2}, \frac{R_i^2}{\rho_i^2})$, where $\frac{R_i^2}{\rho_i^2} \geq \frac{1}{\gamma_i^2} \to \infty$.

By Brakke's compactness theorem, there is a subsequence $i(k)$ such that $\tilde{\mathcal{K}}_{i(k)} \to \mathcal{K}_\infty$, which is also a matching motion $(T_\infty, \mathcal{K}_\infty = \{\mu_{\infty,t}\})$ by Theorem 3. $\mathcal{K}_\infty$ is either eternal or the flow of a topological sphere by Proposition 4, and $0 \in \mathrm{spt}(\mu_{\infty,0})$ by Lemma 4. And satisfying

$$\mu_{\infty,-1}(B_C) < \eta(C)^n$$

or

$$\mu_{\infty,1}(B_C) < \eta(C)^n$$

If $\mathcal{K}_\infty$ is eternal, by choosing $\rho = 1$, we get a contradiction to Proposition 6.

If $\mathcal{K}_\infty$ is the flow of a topological sphere, say $\mu_{\infty,t}$ is a topological sphere that is extinct at time $\tilde{t}$, then By Brakke's regularity Theorem Proposition 1, for large enough $i(k)$, the flow also develops a spherical singularity before time $2\tilde{t}$, contradicting the fact that it does not develop spherical singularities for $t \in (-\frac{R_{i(k)}^2}{\rho_{i(k)}^2}, \frac{R_{i(k)}^2}{\rho_{i(k)}^2})$ where $\frac{R_{i(k)}^2}{\rho_{i(k)}^2} \to \infty$.

□

We now estimate the change of Hausdorff distance in low entropy flows.

Lemma 6. *For $n \geq 3$ and $\delta(n)$ chosen as in the previous section, there is a $C(n, \gamma)$: If $(T, \mathcal{K} = \{\mu_t\})$ is a matching motion in $\mathbb{R}^{n+1} \times [0, t_0]$ with $\lambda(\mathcal{K}) \leq \Lambda_n + \delta(n)$, and μ_t does not develops spherical singularities for $t \in (0, t_0)$, then for any $0 \leq t_1 < t_2 \leq t_0$, we have*

$$\mathrm{dist}_H(\mathrm{spt}(\mu_{t_1}), \mathrm{spt}(\mu_{t_2})) \leq C\sqrt{t_2 - t_1} \tag{3.4.5}$$

Proof. Theorem 7 gives us a $C > 0, \gamma \in (0, 1)$ such that for any $t \in (0, t_0)$, if $0 < \tau < \min(\gamma t, \gamma(t_0 - t))$, then

$$\mathrm{spt}(\mu_{t+\tau}) \text{ is in the } C\sqrt{\tau} \text{ neighborhood of } \mathrm{spt}(\mu_t) \tag{3.4.6}$$

and

$$\mathrm{spt}(\mu_{t-\tau}) \text{ is in the } C\sqrt{\tau} \text{ neighborhood of } \mathrm{spt}(\mu_t) \tag{3.4.7}$$

By replacing t with $t+\tau$, (3.4.7) also gives us

$$\mathrm{spt}(\mu_t) \text{ is in the } C\sqrt{\tau} \text{ neighborhood of } \mathrm{spt}(\mu_{t+\tau}) \tag{3.4.8}$$

Namely

$$\mathrm{dist}_H(\mathrm{spt}(\mu_t), \mathrm{spt}(\mu_{t+\tau})) < C\sqrt{\tau} \tag{3.4.9}$$

Now for any $0 < t_1 < t_2 < t_0$, we can choose η_1, η_2, depending on γ, such that $\frac{1}{2}\gamma < \eta_1, \eta_2 < \gamma$ and $\frac{t_1+t_2}{2} = (1+\eta_1)^{k_1} t_1$ and $(t_0 - t_2)(1+\eta_2)^{k_2} = (t_0 - \frac{t_1+t_2}{2})$ for some $k_1, k_2 \in \mathbb{N}$.

For $i_1 = 0, ..., k_1 - 1$, we have

$$\begin{aligned}
&\mathrm{dist}_H(\mathrm{spt}(\mu_{t_1(1+\eta_1)^{i_1}}), \mathrm{spt}(\mu_{t_1(1+\eta_1)^{i_1+1}})) \\
\leq& C\sqrt{t_1(1+\eta_1)^{i_1+1} - t_1(1+\eta_1)^{i_1}} \\
=& C\sqrt{t_1\eta_1}(\sqrt{1+\eta_1})^{i_1},
\end{aligned}$$

so

$$\begin{aligned}
&\mathrm{dist}_H(\mathrm{spt}(\mu_{\frac{t_1+t_2}{2}}), \mathrm{spt}(\mu_{t_1})) \\
\leq& \Sigma_{i_1=0}^{k_1-1} \mathrm{dist}_H(\mathrm{spt}(\mu_{t_1(1+\eta_1)^{i_1}}), \mathrm{spt}(\mu_{t_1(1+\eta_1)^{i_1+1}})) \\
\leq& \Sigma_{i_1=0}^{k_1-1} C\sqrt{t_1\eta_1}(\sqrt{1+\eta_1})^{i_1} \\
=& C\sqrt{t_1\eta_1}\frac{1-(\sqrt{1+\eta_1})^{k_1}}{1-\sqrt{1+\eta_1}} \\
=& C\frac{\sqrt{\eta_1}}{\sqrt{1+\eta_1}-1}(\sqrt{t_1(1+\eta_1)^{k_1}} - \sqrt{t_1}) \\
\leq& C\frac{\sqrt{\eta_1}}{\sqrt{1+\eta_1}-1}(\sqrt{t_1(1+\eta_1)^{k_1} - t_1}) \\
=& C(\eta_1)(\sqrt{\frac{t_1+t_2}{2} - t_1}) \\
\leq& C(\gamma)(\sqrt{\frac{t_1+t_2}{2} - t_1}).
\end{aligned}$$

.

Similarly for $i_2 = 0, ..., k_2 - 1$, we have

$$\begin{aligned}
&\mathrm{dist}_H(\mathrm{spt}(\mu_{t_0-(t_0-t_2)(1+\eta_2)^{i_2}}), \mathrm{spt}(\mu_{t_0-(t_0-t_2)(1+\eta_2)^{i_2+1}})) \\
\leq &C\sqrt{(t_0-t_2)(1+\eta_2)^{i_2+1} - (t_0-t_2)(1+\eta_2)^{i_2}} \\
= &C\sqrt{(t_0-t_2)\eta_2}(\sqrt{1+\eta_2})^{i_2},
\end{aligned}$$

so

$$\begin{aligned}
&\mathrm{dist}_H(\mathrm{spt}(\mu_{t_2}), \mathrm{spt}(\mu_{\frac{t_1+t_2}{2}})) \\
\leq &\Sigma_{i_2=0}^{k_2-1} \mathrm{dist}_H(\mathrm{spt}(\mu_{t_0-(t_0-t_2)^{i_2}}), \mathrm{spt}(\mu_{t_0-(t_0-t_2)^{i_2+1}})) \\
\leq &\Sigma_{i_2=0}^{k_2-1} C\sqrt{(t_0-t_2)\eta_2}(\sqrt{1+\eta_2}^{i_2}) \\
= &C\sqrt{(t_0-t_2)\eta_2}\frac{1-\sqrt{1+\eta_2}^{k_2}}{1-\sqrt{1+\eta_2}} \\
= &C\frac{\sqrt{\eta_2}}{\sqrt{1+\eta_2}-1}(\sqrt{(t_0-t_2)(1+\eta_2)^{k_2}} - \sqrt{(t_0-t_2)(1+\eta_2)}) \\
\leq &C\frac{\sqrt{\eta_2}}{\sqrt{1+\eta_2}-1}(\sqrt{(t_0-t_2)(1+\eta_2)^{k_2} - (t_0-t_2)(1+\eta_2)}) \\
= &C\frac{\sqrt{\eta_2}}{\sqrt{1+\eta_2}-1}(\sqrt{[t_0-(t_0-t_2)] - [t_0-(t_0-t_2)(1+\eta_2)^{k_2}]}) \\
= &C(\eta_2)(\sqrt{t_2 - \frac{t_1+t_2}{2}}) \\
\leq &C(\gamma)(\sqrt{t_2 - \frac{t_1+t_2}{2}}).
\end{aligned}$$

Now by triangle inequality

$$\begin{aligned}
&\mathrm{dist}_H(\mathrm{spt}(\mu_{t_2}), \mathrm{spt}(\mu_{t_1})) \\
\leq &C(\gamma)(\sqrt{\frac{t_1+t_2}{2} - t_1}) + C(\gamma)(\sqrt{t_2 - \frac{t_1+t_2}{2}}) \\
\leq &C\sqrt{t_2-t_1}
\end{aligned}$$

For the case $t_1 = 0$ or $t_2 = t_0$, since C is independent of t_1, t_2, we can take limits and thus proved the lemma.

□

3.5 Proof of Theorem 1

We are now ready to prove the main theorem. First we use the connectedness of initial hypersurface to get a strong restriction on the behavior of the flows.

Proposition 7. *The $\delta(n)$ can be chosen small enough so that, if $(T, \mathcal{K} = \{\mu_t\}_{t\geq 0})$ is a matching motion in $\mathbb{R}^{n+1}$ with initial data $\mu_0 = \mu_{\Sigma_0}$ being a closed hypersurface, $\lambda(\mathcal{K}) < \Lambda_n + \delta(n)$, then if $\{\mu_t\}$ develops a spherical singularity at space-time point $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, \infty)$, then the flow is extinct at time t_0 at a spherical singularity at x_0.*

Proof. Without loss of generality, we can assume that $x_0 = 0, t_0 = 1$, for otherwise we can do a parabolic translation and dilation to make this happen. The proof is by contradiction

Claim 3.5.1. *If the flow is not extinct at the time $t = 1$ when it first develops a spherical singularity. Then there is a point $y_0 \in B_{4C}(0), y_0 \neq 0$ with Huisken's density $\Theta_{(y_0,t_0)} \geq 1$, where C is the universal constant from the previous Lemma 6.*

Proof. (of Claim 3.5.1) Without the assumption that the point $y_0 \in B_{4C}(0)$, the existence is straightforward since the flow is not extinct yet at time $t = 1$.

Case 1: If $\mathrm{spt}(\mu_0) \subset B_{3C}$, by the Hausdorff estimate Lemma 6, we have

$$\mathrm{dist}_H(\mathrm{spt}(\mu_0), \mathrm{spt}(\mu_1)) \leq C$$

namely, there is a point $y_0 \in \mathrm{spt}(\mu_1)$ such that

$$y_0 \in B_{3C+C}(0) = B_{4C}(0)$$

We can choose $y_0 \neq 0$ because the flow is not extinct at $t = 1$ and $(0, 1)$ is a spherical singularity.

Case 2: If $\mathrm{spt}(\mu_0) = \mu_{\Sigma_0}$ is not contained in B_{3C}, by the connectedness of Σ_0, there is a point $z_0 \in \Sigma_0 \cap (B_{3C} \setminus B_{2C})$. Again by the Hausdorff distance estimate Lemma 6, we have can find a point $y_0 \in \mathrm{spt}(\mu_1)$ such that

$$y_0 \in (B_{3C+C} \setminus B_{2C-C}) = (B_{4C} \setminus B_C)$$

□

If the Proposition were false, there would be a sequence of matching motions $(T_i, \mathcal{K}_i = \{\mu_{i,t}\}_{t\geq 0})$, each of which satisfying that $\mu_{i,0}$ is a closed hypersurface, $\lambda(\mathcal{K}_i) \leq \Lambda_n + \frac{1}{i}$. They all develops a spherical singularity at $(0,1) \in \mathbb{R}^{n+1} \times (0,\infty)$ and has a point $y_i \in \mathrm{spt}(\mu_{i,1}) \cap B_{4C}, y_i \neq 0$ such that $\Theta_{(y_i,1)} \geq 1$.

We use Brakke's compactness theorem to extract a sub-sequential limit flow $\mathcal{K}_\infty = \{\mu_{\infty,t}\}_{t\geq 0}$, which is also a matching motion by Theorem 3. By the lower semi-continuity of entropy, $\lambda(\mathcal{K}_\infty) = \Lambda_n$. By the upper semi-continuity of the Huisken's density, $\Theta_{(0,1)} \geq \Lambda_n$ and $\Theta_{(y_\infty,1)} \geq 1$ for some $y_\infty \neq 0$.

Now this is a contradiction because, by Huisken's monotonicity formula, at an earlier time than $t = 1$, the entropy is strictly greater than Λ_n. Thus we proved the Proposition. □

Proof. (of Theorem 1)

As in [2], we argue by contradiction. Since we cannot rule out the formation of non-compact singularities with low entropy as in [5], we must work in the setting of matching motions instead of smooth flows. According to Proposition 7, for flows starting from a closed hypersurface, the only time when the flow develops a spherical singularity is when it's extinct.

Fix the dimension n, suppose for some $1 > \epsilon > 0$ there are connected closed hypersurfaces $\Sigma_i \subset \mathbb{R}^{n+1}$ with $\lambda(\Sigma_i) \leq \Lambda_n + \frac{1}{2i}$ and so that $\mathrm{dist}_H(\rho\mathbb{S}^n + y, \Sigma_i) > \rho\epsilon > 0$ for any $\rho > 0, y \in \mathbb{R}^{n+1}$.

Claim 3.5.2. *("Triangle" inequality for the scale-invariant Hausdorff distance)*

For each such i, we can find a small graphical perturbation $\tilde{\Sigma}_i$, also connected, such that the level-set flow of $\tilde{\Sigma}_i$ is non-fattening and such that

$$\mathrm{dist}_H(\rho\mathbb{S}^n + \mathbf{y}, \tilde{\Sigma}_i) \geq \frac{\rho\epsilon}{2} > 0$$

for any $\rho > 0, \mathbf{y} \in \mathbb{R}^{n+1}$, and

$$\lambda(\tilde{\Sigma}_i) \leq \Lambda_n + \frac{1}{i}$$

Proof. (of Claim 3.5.2)

By Proposition 2, we can choose $\tilde{\Sigma}_i$ being non-fattening and such that $\mathrm{dist}_H(\Sigma_i, \tilde{\Sigma}_i)$ is arbitrary small because the Hausdorff distance is bounded by the C^0 graphical norm.

First, for $\rho < \frac{1}{16}\mathrm{diam}(\Sigma_i)$, if $\tilde{\Sigma}_i$ is chosen so that $\mathrm{dist}_H(\Sigma_i, \tilde{\Sigma}_i) < \frac{1}{16}\mathrm{diam}(\Sigma_i)$, we already have $\mathrm{dist}_H(\rho\mathbb{S}^n + \mathbf{y}, \tilde{\Sigma}_i) \geq \frac{\rho\epsilon}{2}$ for any $\mathbf{y} \in \mathbb{R}^{n+1}$. This is because $\mathrm{diam}(\tilde{\Sigma}_i) > \frac{14}{16}\mathrm{diam}(\Sigma_i) > 14\rho$, so cannot lie in the $\frac{\rho\epsilon}{2}$ neighborhood of any round n-sphere of radius ρ in $\mathbb{R}^{n+1}$. Namely, for any $0 < \rho < \frac{1}{16}\mathrm{diam}(\Sigma_i), \mathbf{y} \in \mathbb{R}^{n+1}$,

$$\mathrm{dist}_H(\tilde{\Sigma}_i, \rho\mathbb{S}^n + \mathbf{y}) \geq \frac{\rho\epsilon}{2}$$

Next, for $\rho \geq \frac{1}{16}\mathrm{diam}(\Sigma_i)$, we will choose $\tilde{\Sigma}_i$ so that $\mathrm{dist}_H(\tilde{\Sigma}_i, \Sigma_i) < \frac{1}{32}\mathrm{diam}(\Sigma_i)\epsilon < \frac{\rho\epsilon}{2}$. Then for any $\mathbf{y}$,

$$\begin{aligned}
&\mathrm{dist}_H(\tilde{\Sigma}_i, \rho\mathbb{S}^n + \mathbf{y}) \\
\geq &\mathrm{dist}_H(\Sigma_i, \rho\mathbb{S}^n + \mathbf{y}) - \mathrm{dist}_H(\tilde{\Sigma}_i, \Sigma_i) \\
\geq &\rho\epsilon - \frac{\rho\epsilon}{2} \\
= &\frac{\rho\epsilon}{2}
\end{aligned}$$

By the lower semi-continuity of entropy, there is a $\delta_i > 0$ such that if the $\tilde{\Sigma}_i$ chose as a graph u_i over Σ_i with $||u_i||_{C^0(\Sigma_i)} < \delta$, then $\lambda(\tilde{\Sigma}_i) < \lambda(\Sigma_i) + \frac{1}{2i} \leq \Lambda_n + \frac{1}{i}$.

Thus, we can choose $\tilde{\Sigma}_i$ according to Proposition 2 so that it's a graph u_i over Σ_i with $||u_i||_{C^0(\Sigma_i)} < \min(\frac{1}{16}\mathrm{diam}(\Sigma_i), \frac{1}{32}\mathrm{diam}(\Sigma_i)\epsilon, \delta_i)$, and proved the claim.

$\square$

Now we have a sequence of matching motions $(T_i, \mathcal{K}_i = \{\mu_{i,t}\})$, with $\mu_{i,0} = \mu_{\tilde{\Sigma}_i}$. Since $\tilde{\Sigma}_i$ are all closed hypersurfaces, the flow must become extinct in finite time t_i (by the avoidance principle), at a round spherical singularity by Proposition 3.

Next we parabolically rescale the flows $(T_i, \mathcal{K}_i)$ by factors $\frac{1}{\sqrt{t_i}}$ and translate the extinction point (x_i, t_i) to the space-time origin to get $(\tilde{T}_i, \tilde{\mathcal{K}}_i)$, $\tilde{\mathcal{K}}_i = D_{\frac{1}{\sqrt{t_i}}}(\mathcal{K}_i - (x_i, t_i)) = \{\tilde{\mu}_{i,t}\}$. The flows $\tilde{\mathcal{K}}_i$ has non-empty support and no spherical singularities for time $t \in [-1, 0)$.

By Lemma 6, after throwing out small values of i, there is a $\tau \in (-1, -\frac{1}{2})$, independent of i, so that

$$\mathrm{dist}_H(\mathrm{spt}(\tilde{\mu}_{i,-1}), \mathrm{spt}(\tilde{\mu}_{i,\tau})) + \mathrm{dist}_H(\sqrt{2n}\mathbb{S}^n, \sqrt{-2n\tau}\mathbb{S}^n) < \frac{1}{8}\epsilon \tag{3.5.1}$$

By Brakke's compactness theorem, there is a subsequence $i(k)$ such that $\tilde{\mathcal{K}}_{i(k)}$ converges to a limit flow $\mathcal{K}_\infty = \{\mu_{\infty,t}\}_{t\in[-1,0]}$, which is a matching motion $(T_\infty, \mathcal{K}_\infty)$ by Theorem 3, and $\lambda(\mathcal{K}_\infty) = \Lambda_n$.

The uniqueness Theorem 5 tells us this limit flow is the regular flow of round n-sphere.

Now we apply Proposition 1 with $\epsilon = \frac{\tau+1}{2}$ and the limit flow being the regular round sphere. For sufficiently large $i(k)$, by connectedness of $\tilde{\Sigma}_i$, $\tilde{\mathcal{K}}_i$ is sufficiently close to the regular flow of sphere for $t \in (-1+\epsilon, 0)$, namely we can choose $i(k_0)$ such that

$$\mathrm{dist}_H(\mathrm{spt}(\tilde{\mu}_{i(k_0),\tau}), \sqrt{-2n\tau}\mathbb{S}^n) < \frac{1}{4}\epsilon \tag{3.5.2}$$

By (3.5.1) and triangle inequality, we have

$$\operatorname{dist}_H(\operatorname{spt}(\tilde{\mu}_{i,-1}), \sqrt{2n}\mathbb{S}^n) < \frac{1}{8}\epsilon + \frac{1}{4}\epsilon < \frac{1}{2}\epsilon \tag{3.5.3}$$

That is

$$\operatorname{dist}_H(\tilde{\Sigma}_i, \sqrt{2nt_i}\mathbb{S}^n + \mathbf{x_i}) < \frac{1}{2}\sqrt{t_i}\epsilon < \frac{1}{2}\sqrt{2nt_i}\epsilon \tag{3.5.4}$$

contradicting our choice of $\tilde{\Sigma}_i$ in Claim 3.5.2, and proves the theorem.

□

4

No disconnection in low entropy level-set flow

4.1 Main result

When $n = 1$, it follows from Gage-Hamilton [21] and Grayson [23] that the flow disappears when it becomes singular. In particular, the flow remains connected until it disappears. In contrast, when $n > 1$, non-degenerate neck-pinch examples show that there are flows that become singular without disappearing. In these examples, the level set flow disconnects after the neck-pinch singularity. In [5], the first author and L. Wang showed that, when $n = 2$ and the entropy of the initial surface is small enough, then the flow also disappears at its first singularity. This result makes use of a classification of singularity models in $\mathbb{R}^3$ of low entropy from [5] and whether such a classification exists in higher dimension is unknown.

In the second topic of this thesis here we show that when $n = 3$ and the initial hypersurface is closed, connected and of low entropy, then even if the flow forms a singularity before it disappears, its level set flow remains connected until its extinction time.

Theorem 8. *Let $\Sigma \subset \mathbb{R}^4$ be a closed, connected hypersurface and let $\{\Gamma_t\}_{t\in[0,T]}$ be the level set flow with initial condition $\Gamma_0 = \Sigma$ and extinction time T. If $\lambda(\Sigma) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, then, for all $t \in [0, T]$, Γ_t is connected. Moreover, if $W[t] = \mathbb{R}^4 \setminus \Gamma_t$, then $W[t]$ has at most two connected components for all $t \in [0, T]$.*

Remark 6. *A technical feature of the level set flow is that it may "fatten", i.e., develop non-empty interior. If this occurs in Theorem 8, then there will be a $T_0 \in [0, T)$ so that $W[t]$ has two components* *for $t \in [0, T_0)$ and one component for $t \in [T_0, T]$.*

In [35], the second author showed that, for mean curvature flows of low entropy, if the flow reaches the point x_0 at time t_0, then, the flow remains near x_0 after t_0 until it disappears. This is a forward in time analog of the standard, unconditional, clearing out lemma – e.g., [16, Theorem 3.1] – that says that if the flow reaches x_0 at time t_0, then the flow must be near x_0 at earlier times. Theorem 8 allows us to sharpen the result from [35] and prove the forward clearing out lemma in $\mathbb{R}^4$ with the optimal upper bound on the entropy.

Corollary 2. *There exist uniform constants $C > 1$ and $\eta > 0$, so that if $\{M_t\}_{t\in[0,T]}$ is a a non-fattening level set flow in $\mathbb{R}^4$ that starts from a smooth closed hypersurface $M_0 \subset \mathbb{R}^4$ with $\lambda(M_0) < \lambda(\mathbb{S}^2 \times \mathbb{R})$, $x_0 \in M_{t_0}$ and $M_{t_0+R^2} \neq \emptyset$, then for all $\rho \in (0, \frac{R}{2C})$,*

$$\mathcal{H}^3(B_\rho(x_0) \cap M_{t_0+C^2\rho^2}) \geq \eta\rho^3.$$

Here $\mathcal{H}^3$ denotes the 3-dimensional Hausdorff measures.

Remark 7. *The entropy assumption can be seen to be sharp by considering the translating bowl soliton in $\mathbb{R}^4$ and, in the closed setting, by considering a sequence of unit spheres at increasing distance from one another and joined by a thin tube.*

4.2 Strong canonical boundary motions

Remember that associated to each $E \subset \mathbb{R}^{n+1} \times \mathbb{R}$ of locally finite perimeter, there is a unique $(n+2)$-dimensional integral current $[E] \in \mathbf{I}^{loc}_{n+2}(\mathbb{R}^{n+1} \times \mathbb{R})$. Similarly, given an oriented codimension-k submanifold $\Sigma \subset \mathbb{R}^{n+1} \times \mathbb{R}$ there is a unique $[\Sigma] \in \mathbf{I}^{loc}_k(\mathbb{R}^{n+1} \times \mathbb{R})$. If $\partial^* E$ is the reduced boundary of E, then $[\partial^* E] = \partial[E] \in \mathbf{I}^{loc}_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R})$. As such, there is an integer $(n+1)$-rectifiable Radon measure $\mathcal{H}^n \lfloor \partial^* E$ – see [25] for details.

We extend the notion of canonical boundary motion from [3] – see also [25, 4]. These flows are special cases of flows introduced by Ilmanen in [25] that synthesis the level set flow and Brakke flow in a natural way and are key to our approach.

Definition 6. *A* canonical boundary motion *is a triple $(E_0, E, \mathcal{K})$ consisting of an open bounded set $E_0 \subset \mathbb{R}^{n+1} \times \{0\}$ with ∂E_0 a smooth closed hypersurface, an open bounded set $E \subset \mathbb{R}^{n+1} \times [0, \infty)$ of finite perimeter and a Brakke flow $\mathcal{K} = \{\mu_t\}_{t\geq 0}$ so:*

1. *$E = \{(x,t) : u(x,t) > 0\}$, where u solves equation (3.2.5) with u_0 chosen so $E_0 = \{x : u_0(x) > 0\}$ and $\partial E_0 = \{x : u_0(x) = 0\}$;*

2. *The level set flow of ∂E_0 is non-fattening;*

3. *For $t \geq 0$, each $E_t = \{x : (x,t) \in E\}$ is of finite perimeter and $\mu_t = \mathcal{H}^n \lfloor \partial^* E_t$.*

If, in addition,

4. $$\{u = 0\} = \overline{\partial^* E} \text{ in } \mathbb{R}^{n+1} \times (0, \infty)$$

where u is from Item (1), then $(E_0, E, \mathcal{K})$ is a strong canonical boundary motion.

Remark 8. *Observe, $\{u > 0\} = E \subset \bar{E} \subset \{u \geq 0\}$ for a canonical boundary motion and $\bar{E} = \{u \geq 0\}$ for a strong canonical boundary motion. If $\Gamma_t = \{x \in \mathbb{R}^{n+1} | u(x,t) = 0\}$, then $\{\Gamma_t\}_{t \geq 0}$ is the level set flow of $\Gamma_0 = \Sigma$ and is non-fattening. Clearly, $\partial E_t \subset \Gamma_t$, but equality need not hold – even for strong canonical boundary motions.*

By [25, 11.4], for a E_0 with the property that the level set flow of ∂E_0 is non-fattening, there are E and $\mathcal{K}$ so $(E_0, E, \mathcal{K})$ is a canonical boundary motion. In general, the non-fattening condition is not enough to ensure the existence of a strong canonical boundary motion, however, in [25, 12.11], Ilmanen shows such existence for "generic" E_0.

Finally, we introduce the following notation for a level set flow $\{\Gamma_t\}_{t \geq 0}$ in $\mathbb{R}^{n+1}$, $n \geq 1$,

$$\begin{aligned} W[t] &= \mathbb{R}^{n+1} \setminus \Gamma_t \\ W[s,r] &= \{(x,t) | x \in (\mathbb{R}^{n+1} \setminus \Gamma_t), s \leq t \leq r\} = \bigcup_{t \in [s,r]} W[t] \\ n(t) &= \#\{\text{connected components of } W[t]\} \in \mathbb{N} \cup \{\infty\}. \end{aligned}$$

As Γ_t is compact and $n \geq 1$, there is exactly one unbounded component of $W[t]$, denoted by $W^-[t]$. Let $W^+[t] = W[t] \backslash W^-[t]$ be the bounded components and set

$$W^{\pm}[s,r] = \bigcup_{t \in [s,r]} W^{\pm}[t].$$

4.3 Proof of the result for strong canonical boundary motions

Let's first record some elementary preliminary results.

The first is an elementary topological result – we include a proof for the sake of completeness.

Lemma 7. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a compact set. If $\mathbb{R}^{n+1}\backslash\Gamma$ has exactly two components, $W^{\pm}$, and $\Gamma = \partial W^{\pm}$, then Γ is connected.*

Proof. Suppose that Γ is not connected. Let K be one component of Γ and $K' = \Gamma\backslash K \neq \emptyset$. Observe that both K and K' are compact and so there is a $r > 0$ so that $T_r(K)\cap T_r(K') = \emptyset$ and, hence, $T_r(\Gamma)$ is not connected. Let $\hat{W}^{\pm} = W^{\pm} \cup T_r(\Gamma)$. Clearly, $\hat{W}^{\pm}$ are open sets with $\hat{W}^+ \cap \hat{W}^- = T_r(\Gamma)$. For each $x \in \Gamma$, $W^{\pm} \cap B_r(x) \neq \emptyset$ as $\Gamma = \partial W^{\pm}$. As the union of intersecting connected sets is connected, $W^{\pm} \cup B_r(x)$ is connected. It readily follows that both $\hat{W}^-$ and $\hat{W}^+$ are connected. Finally, by the Mayer-Vietoris long exact sequence for reduced homology, as $\mathbb{R}^{n+1} = \hat{W}^+ \cup \hat{W}^-$ is simply connected and both $\hat{W}^{\pm}$ are connected, $T_r(\Gamma) = \hat{W}^+ \cap \hat{W}^-$ must be connected. This contradicts our choice of r and proves the lemma. □

Another elementary fact is that the level set flow remains connected up to and including its first disconnection time.

Lemma 8. *Let $\{\Gamma_t\}_{t\in[0,T]}$ be a level set flow of compact sets in $\mathbb{R}^{n+1}$. If Γ_t is connected for $t \in [0, t_0)$, then Γ_{t_0} is connected.*

Proof. By the definition and basic properties of level set flow $\lim_{t\to t_0^-} \Gamma_t = \Gamma_{t_0}$ in Hausdorff distance. On the one hand, by the avoidance principle,

$$\Gamma_{t_0} \subset T_{\sqrt{4n(t_0-t)}}(\Gamma_t).$$

On the other, as the space-time track of the level set flow, $\mathbb{R}^{n+1} \times [0, T]\backslash W[0, T]$, is closed and Γ_{t_0} is compact, for every $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < t_0 - t < \delta$, then $\Gamma_t \subset T_\epsilon(\Gamma_{t_0})$. Hence, if Γ_{t_0} is disconnected, then for $t < t_0$ close enough to t_0, Γ_t is disconnected, proving the claim. □

The next result summarizes and extends of [4] and provides a description of the regularity properties of strong canonical boundary motions flows in $\mathbb{R}^4$ of low entropy.

Proposition 8. *Let $(E_0, E, \mathcal{K} = \{\mu_t\}_{t\geq 0})$ be a strong canonical boundary motion in $\mathbb{R}^4$. Suppose the flow has extinction time T and $\Sigma_0 = \partial E_0$ satisfies $\lambda(\Sigma_0) < \Lambda_2$.*

1. *For each $t \in [0, T)$, there are a finite, possibly empty, set of points $x_1, \ldots, x_{m(t)} \in \mathbb{R}^4$ so that $\mu_t = \mathcal{H}^3\lfloor\Sigma_t$ where Σ_t is a hypersurface in $\mathbb{R}^4\backslash\{x_1, \ldots, x_m\}$.*

2. *For an open dense subset $I \subset [0, T]$, if $t \in I$, then $\mu_t = \mathcal{H}^3\lfloor\Sigma_t$ where Σ_t is a closed hypersurface.*

3. *Let $(x_0, t_0) \in \mathbb{R}^4 \times (0, T]$ be a point at which $\mathcal{K}$ has positive Gaussian density, if $\{\nu_t\}_{t\in\mathbb{R}} = \mathcal{T} \in \mathrm{Tan}_{(x_0,t_0)}\mathcal{K}$, then $\nu_{-1} = \mathcal{H}^3 \lfloor \Upsilon$ where Υ is a smooth self-shrinker and either Υ is closed or it is asymptotically conical. Moreover, whichever holds depends only on (x_0, t_0) and not on the choice of tangent flow.*

4. *For each $(x_0, t_0) \in \mathbb{R}^4 \times (0, T]$ for which $\mathrm{Tan}_{(x_0,t_0)}\mathcal{K}$ contains an asymptotically conical shrinker, there is an $R_0 = R_0(x_0, t_0, \partial E_0) > 0$ so that for all $R \in (0, R_0]$*

$$\Sigma_{t_0}(x_0, R) = \mathrm{spt}(\mu_{t_0}) \cap B^*_R(x_0) = \Sigma_{t_0} \cap B^*_R(x_0) = \partial E_{t_0} \cap B^*_R(x_0) = \partial^* E_{t_0} \cap B^*_R(x_0),$$

is a connected hypersurface that divides $B^*_R(x_0)$ into two components, one contained in E_t and one disjoint from it. Here $B^*_R(x_0) = B_R(x_0)\backslash\{x_0\}$.

Proof. Note first that as $(E_0, E, \mathcal{K})$ is a strong canonical boundary motion, $(E, \mathcal{K})$ is a canonical boundary motion in the sense of [4] – see Theorem 2.3 and the discussion at the beginning of Section 4 of [4]. As such, Items (1) and (2) are both immediate consequences of [4, Theorem 4.3] – see [4, Corollary 4.4] and the proof of [4, Theorem 4.5] for details. Item (3) follows from [4, Proposition 4.1 and Lemma 4.2].

It remains to show Item (4). First, set $\epsilon_0 = \Lambda_2 - \lambda(\partial E_0) > 0$. Next observe that if (x_0, t_0) is a singular point of $\mathcal{K}$, then, by hypothesis it is a non-compact singularity and so by [4, Theorem 4.2(2)], there is a $\alpha = \alpha(\epsilon_0) > 0$ and a $\rho_0 = \rho_0(x_0, t_0) > 0$ so that for all $(\rho, t) \in (0, \rho_0) \times (t_0 - \rho^2, t_0 + \rho^2)$,

$$A_t(x_0, t_0, \rho) = \Sigma_t \cap \left(B_{2\alpha\rho}(x_0)\backslash\bar{B}_{\frac{1}{2}\alpha\rho}(x_0)\right) = \mathrm{spt}(\mu_t) \cap \left(B_{2\alpha\rho}(x_0)\backslash\bar{B}_{\frac{1}{2}\alpha\rho}(x_0)\right)$$

is a connected non-empty hypersurface that is proper in $B_{2\alpha\rho}(x_0)\backslash\bar{B}_{\frac{1}{2}\alpha\rho}(x_0)$. The same is true if (x_0, t_0) is not a singular point as then $\mathrm{Tan}_{(x_0,t_0)}\mathcal{K}$ consists of a static hyperplane. For $\rho \in (0, \rho_0)$, let

$$A(x_0, t_0, \rho) = \bigcup_{t\in(t_0-\rho^2, t_0+\rho^2)} A_t(x_0, t_0, \rho) \times \{t\}$$

this is a connected non-empty hypersurface that is proper in the hollow space-time cylinder

$$C(x_0, t_0, \rho) = \left(B_{2\alpha\rho}(x_0)\backslash\bar{B}_{\frac{1}{2}\alpha\rho}(x_0)\right) \times (t_0 - \rho^2, t_0 + \rho^2).$$

Clearly, $A_t(x_t, t_0, \rho) = A(x_0, t_0, \rho) \cap \{x_5 = t\}$ and this intersection is transverse.

By Item (3) of the definition of canonical boundary motion, $\mathrm{spt}(\mu_t) = \overline{\partial^* E_t}$, and so

$$A(x_0, t_0, \rho) = \overline{\partial^* E} \cap C(x_0, t_0, \rho).$$

As $A(x_0, t_0, \rho)$ is smooth, every point is in the reduced boundary and so

$$A(x_0, t_0, \rho) = \partial^* E \cap C(x_0, t_0, \rho).$$

Hence, by Item (4) of the definition of a strong canonical boundary motion,

$$A(x_0, t_0, \rho) = \partial^* E \cap C(x_0, t_0, \rho) = \overline{\partial^* E} \cap C(x_0, t_0, \rho) = \partial E \cap C(x_0, t_0, \rho).$$

Together with the fact that that $A(x_0, t_0, \rho)$ meets $\{x_5 = t\}$ transversally, this means

$$A_{t_0}(x_0, t_0, \rho) = \partial^* E_{t_0} \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0)\right) = \partial E_{t_0} \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0)\right).$$

Set $R_0 = 2\alpha\rho_0$ and, for any $R \in (0, R_0)$, let

$$\Sigma_{t_0}(x_0, R) = \bigcup_{i=0}^{\infty} A_{t_0}(x_0, t_0, 2^{-1} R).$$

By the above, $\Sigma_{t_0}(x_0, R)$ is a connected non-empty hypersurface proper in $B_R^*(x_0)$ and, moreover,

$$\Sigma_{t_0}(x_0, R) = \partial^* E_{t_0} \cap B_R^*(x_0) = \partial E_{t_0} \cap B_R^*(x_0) = \mathrm{spt}(\mu_{t_0}) \cap B_R^*(x_0) = \Sigma_{t_0} \cap B_R^*(x_0).$$

Finally, as $\Sigma_{t_0}(x_0, R)$ is connected, non-empty and proper in $B_R^*(x_0)$, $B_R^*(x_0) \setminus \Sigma_{t_0}(x_0, R)$ has two components. On the one hand, $\Sigma_{t_0}(x_0, R) \subset \partial E_{t_0}$ implies at least one of these is a subset of E_t. On the other, $\Sigma_{t_0}(x_0, R) \subset \partial^* E_{t_0}$ means the other is disjoint from E_{t_0}. □

Next we use the above regularity properties to strengthen the relationships between the level set flow and its interior for strong canonical boundary motions of low entropy – compare with Remark 8.

Proposition 9. *Let $(E_0, E, \mathcal{K} = \{\mu_t\}_{t \geq 0})$ be a strong canonical boundary motion in $\mathbb{R}^4$ with $\lambda(\partial E_0) < \Lambda_2$ and let $\{\Gamma_t\}_{t \in [0,T]}$ be the level set flow with $\Gamma_0 = \Sigma$. If there is a $t_0 \in (0, T]$, so for all $(x, t) \in \mathbb{R}^4 \times (0, t_0]$, $\mathrm{Tan}_{(x,t)}\mathcal{K}$ is either trivial or consists of only asymptotically conical* *tangent flows, then for all $s \in [0, t_0]$,*

$$\Gamma_s = \mathrm{spt}(\mu_s) = \partial E_s = \partial(\mathbb{R}^4 \backslash \bar{E}_s).$$

If, in addition, Γ_s is connected, then $E_s = W^+[s]$ and $\Gamma_s = \partial W^{\pm}[s]$.

Proof. As the level set flow is the biggest flow, $\mathrm{spt}(\mu_t) \subset \Gamma_t$ – see [25, 10.7]. Pick a $s \in (0, t_0]$ and a $x_0 \in \Gamma_s$. Let $\mathcal{T} \in \mathrm{Tan}_{(x_0,s)}\mathcal{K}$ be a tangent flow to $\mathcal{K}$ at the point (x_0, s). By Item (4) of the definition of strong canonical boundary motion, $(x_0, s) \in \overline{\partial^* E}$. Hence, there is a sequence $(x_i, s_i) \in \partial^* E$ with $s_i > 0$ and $\lim_{i\to\infty}(x_i, s_i) = (x_0, s)$. As $(x_i, s_i) \in \partial^* E$, the Gaussian density of $\mathcal{K}$ at (x_i, s_i) is at least 1 and so, by the upper semicontinuity property of Gaussian density, the Gaussian density of $\mathcal{K}$ at (x_0, s) is positive and so $\mathcal{T}$ is non-trivial. Hence, by Item (3) of Proposition 8 and the hypothesis, $\mathcal{T} = \{\nu_t\}_{t\in\mathbb{R}}$ is asymptotically conical.

Thus, Item (4) of Proposition 8 implies that there is a $R_0 > 0$ so for all $R \in (0, R_0)$, $\mathrm{spt}(\mu_s) \cap B^*_R(x_0)$ is non-trivial. As $\mathrm{spt}(\mu_s)$ is closed, this means that $x_0 \in \mathrm{spt}(\mu_s)$ and hence, $\mathrm{spt}(\mu_t) = \Gamma_t$ for all $t \in (0, t_0]$ proving the first equality. To see the second equality, first note that, by definition, $\partial E_s \subset \Gamma_s$. Now suppose that $x_0 \in \Gamma_s$. By what we have already shown we know that $x_0 \in \mathrm{spt}(\mu_s)$ and Item (4) of Proposition 8 holds at (x_0, s). Hence, there is a $R_0 > 0$ so for all $R \in (0, R_0)$,

$$\mathrm{spt}(\mu_s) \cap B^*_R(x_0) = \partial E_s \cap B^*_R(x_0)$$

and this intersection is non-empty. As the topological boundary of a set is closed, $x_0 \in \partial E_s$ and so $\Gamma_s = \partial E_s$, proving the second equality. As the other component given by Item (4) of Proposition 8 is disjoint from $\bar{E}_s$, the same argument proves the third equality.

To complete the proof, first observe that, by definition, $E_s \subset W^+[s]$ and $\partial E_s \subset \partial W^+[s] \subset \Gamma_s$. As $\partial E_s = \Gamma_s$ this immediately implies $\Gamma_s = \partial W^+[s]$. Similarly, by definition $\partial W^-[s] \subset \Gamma_s$, and, for any $x \in \partial W^-[s]$. Hence, Item (4) of Proposition 8 implies that there is an $R > 0$ so that $B^*_R(x) \cap \Gamma_s$ divides $B^*_R(x)$ into exactly two components, $U^{\pm}(x)$, with $\partial U^{\pm}(x) \cap B^*_R(x) = \Gamma_s \cap B^*_R(x)$ and so that, up to relabeling, $U^+(x) \subset E_s$ and $U^-(x) \cap E_s = \emptyset$. As $x \in \partial W^-[s]$ and $W^-[s] \cap E_s = \emptyset$, $U^-(x) \subset W^-[s]$ and so $\partial W^-[s] \cap B^*_R(x) = \Gamma_s \cap B^*_R(x)$. Hence, as $x \in \partial W^-[s] \subset \Gamma_s$, $B_R(x) \cap \Gamma_s \subset \partial W^-[s]$ and so $\partial W^-[s]$ is an open non-empty subset of Γ_s. As Γ_s is assumed to be connected, this means $\Gamma_s = \partial W^-[s]$. Finally, let $\Omega = W^+[s] \backslash E_s$. As $\partial E_s = \Gamma_s = \partial W^+[s]$, $\partial\Omega \subset \Gamma_s$. For each $x \in \Gamma_s$, Item (4) of Proposition 8, implies that, for R sufficiently small, $B_R(x) \backslash \Gamma_s$ consists of two components one disjoint from E_s and one contained in E_s. As $B_R(x) \cap W^-[s] \neq \emptyset$ the component disjoint from E_s is contained in $W^-[s]$ and so is disjoint from Ω. Likewise, the component contained in E_s is disjoint from Ω by construction. Hence, $\Omega \cap B_R(x) = \emptyset$ and so $x \notin \partial\Omega$. As x was arbitrary, this means $\partial\Omega = \emptyset$ which implies $\Omega = \emptyset$. That is, $E_s = W^+[s]$. $\square$

We use the preceding results and ideas from [37] to show that strong canonical boundary motions remain connected until they disappear. That is, we show Theorem 8 for strong canonical boundary motions.

Proposition 4.3.1. *Let $(E_0, E, \mathcal{K} = \{\mu_t\}_{t\geq 0})$ be a strong canonical boundary motion in $\mathbb{R}^4$ with E_0 connected and $\lambda[\partial E_0] < \Lambda_2$. If $\{\Gamma_t\}_{t\in[0,T]}$ is the level set flow with $\Gamma_0 = \partial E_0$ and extinction time T, then Γ_t is connected and $n(t) = 2$ for all $t \in [0, T)$.*

Proof. As E_0 is connected and bounded and $\partial E_0 = \Sigma$ is compact, $W^+[0] = E_0$. As Σ is a hypersurface, there is a $\delta > 0$ so that Γ_t is a smooth flow for $t \in [0, \delta]$ and so Γ_t is connected, $n(t) = 2$ and $W[t] = E_t$ for $t \in [0, \delta]$. Let

$$t_{dis} = \sup\{t \in (0, T) | n(s) = 2 \text{ and } \Gamma_s \text{ is connected for all } 0 \leq s < t\}$$

be the first possible disconnection time. Clearly, $t_{dis} > \delta$ and if $t_{dis} = T$, then we are done. In what follows we suppose $t_{dis} < T$ and derive a contradiction.

First observe that, by construction, t_{dis} must be a singular time, but not the extinction time of the flow. As such, for any $(x, t_0) \in \mathbb{R}^4 \times (0, t_{dis}]$, for which $\mathcal{K}$ has positive Gaussian density all tangent flows to $\mathcal{K}$ at (x, t_0) are asymptotically conical. Indeed, by Proposition 8, if a tangent flow at (x, t_0) was closed, then, as Γ_t was connected for $t < t_0 < t_{dis}$, for $t < t_0$ and t close enough to t_0, $\mathrm{spt}(\mu_t)$ would also be a closed connected hypersurface. This would imply that the whole flow becomes extinct at t_0, contradicting the fact that $t_{dis} < T$ is not the extinction time.

By Lemma 8 and the definition of t_{dis}, Γ_t is connected for all $t \in [0, t_{dis}]$. Hence, by Proposition 9, for all $t \in [0, t_{dis}]$, $\Gamma_t = \mathrm{spt}(\mu_t) = \partial W^\pm[t]$. We conclude that $n(t_{dis}) = 2$. Indeed, if $n(t_{dis}) \geq 3$, then as $W^-[t]$ is connected, there is a component, Ω, of $W^+[t_{dis}]$ so $\Omega' = W^+[t_{dis}]\backslash\Omega$ is non-empty. As $E_{t_{dis}} = W^+[t_{dis}] = \Omega \cup \Omega'$, $\Omega \cap \Omega' = \emptyset$ and Ω, Ω' are both open, $\Gamma_{t_{dis}} = \partial E_{t_{dis}} = \partial\Omega \cup \partial\Omega'$. Hence, as $\Gamma_{t_{dis}}$ is connected, there is an $x \in \partial\Omega \cap \partial\Omega'$. By Item (4) of Proposition 8, there is an $R > 0$ so that $B^*_R(x) \cap E_{t_{dis}}$ has exactly one non-empty component, namely, $B^*_R(x) \cap \Omega = B^*_R(x) \cap \Omega'$. This contradicts $\Omega \cap \Omega' = \emptyset$ and implies $n(t_{ids}) = 2$.

Next observe that there is a $\delta_0 > 0$ so that there are no compact singularities in the time interval $[t_{dis}, t_{dis} + \delta_0]$. Indeed, by Item (3) of Proposition 8, singularities are compact if and only if they are collapsed. Furthermore, by [3, Proposition 4.10] the limit of collapsed singularities is also a collapsed singularity. Hence, if there is no such δ_0, then, by Proposition 8, the flow would have a compact singularity at $t = t_{dis}$ and this has already been ruled out.

For each $t \in [0, T]$, let $\mathcal{C}[t]$ be the set of components of $W[t]$. By [37, Theorem 5.2], for any $0 \leq t < s \leq T$, there is a well-defined map $\pi_{s,t} : \mathcal{C}[s] \to \mathcal{C}[t]$ given by $\pi_{s,t}(\Omega_s) = \Omega_t$ if and only if there is a time-like continuous path in $W[t, s]$, connecting a point in $\Omega_s \times \{s\}$ to a point in $\Omega_t \times \{t\}$. Using Proposition 9 and the fact that there are no compact singularities in $[t_{dis}, t_{dis} + \delta_0]$, it is clear that for all $t_{dis} \leq t < s \leq t_{dis} + \delta_0$ the map $\pi_{s,t}$ is surjective. Hence, $n(t)$ is a non-decreasing function on $[t_{dis}, t_{dis} + \delta_0]$. By Item (2) of Proposition 8 and Proposition 9, there is a $s \in (t_{dis}, t_{dis} + \delta_0)$ so that Γ_s is a smooth closed hypersurface and so $n(s) < \infty$. Hence, setting $k = \inf \{n(t)|t \in (t_{dis}, s)\}$, the definition of t_{dis} and the monotonicity of $n(t)$ implies that $2 \leq k < \infty$ and there is a $\delta_1 \in (0, \delta_0)$ so that $n(t) = k$ for $t \in (t_{dis}, t_{dis} + \delta_1)$. For any $t_{dis} < t < s < t_{dis} + \delta_1$, the fact that $\pi_{s,t}$ is surjective and $n(s) = n(t) = k$ is finite implies that $\pi_{s,t}$ is a bijection.

We claim that $k > 2$. Indeed, for any $t \in (t_{dis}, t_{dis} + \delta_1)$, if $n(t) = 2$, then $W^+[t] = E_t$. This is because there always exactly one unbounded component, $W^-[t]$, whereas E_t is always a bounded component of $W^+[t]$. By Proposition 9, as there are no compact singularities in $[0, t_{dis} + \delta_1]$, $\Gamma_t = \partial W^{\pm}[t]$ and so Lemma 7 implies Γ_t is connected. Hence, if $k = 2$, then not only is $n(t) = 2$ in $(t_{dis}, t_{dis} + \delta_1)$, but Γ_t is connected. This contradicts the definition of t_{dis} and so we must have $k > 2$.

Now choose any $t' \in [t_{dis}, t_{dis} + \delta_1)$. As $n(t') = k > 2 = n(t_{dis})$, the pigeonhole principle implies that there must be two points x_1, x_2 from different components, Ω'_1, Ω'_2 of $W[t']$ so that $(x_1, t'), (x_2, t')$ are each connected via time-like paths in $W[t_{dis}, t']$ to the same component of $W[t_{dis}] \times \{t_{dis}\}$. Label the two paths, $p_1(s), p_2(s)$, so that $p_1(1) = (x_1, t'), p_2(1) = (x_2, t')$. As $p_1(0), p_2(0)$ are in the same component of $W[t_{dis}] \times \{t_{dis}\}$, there is a path p_3 in $W[t_{dis}]$ so that $(p_3(0), t_{dis}) = p_1(0), (p_3(1), t_{dis}) = p_2(0)$. By the avoidance principle, there is a universal constant $C > 0$ so that if $B_r(y) \cap \Gamma_{t_{dis}} = \emptyset$, then $(y, t) \subset W[t]$ for any $t \in [t_{dis}, t_{dis} + Cr^2]$. As $p_3([0, 1])$ is compact, we can choose $0 < r_0 < \mathrm{dist}(p_3[0, 1], \Gamma_{t_{dis}})$. Hence, the avoidance principle gives

$$p_3([0, 1]) \times [t_{dis}, t_{dis} + Cr_0^2] \subset W[t_{dis}, t_{dis} + Cr_0^2]$$

As such, if $\delta_2 = \min\left\{\frac{t'-t_{dis}}{2}, Cr_0^2, \delta_1\right\}$, then for any $t \in (t_{dis}, t_{dis} + \delta_2)$, $(x_1, t'), (x_2, t')$ can also be connected via time-like paths in $W[t, t']$ to the same components of $W[t]$. That is, $\pi_{t',t}(\Omega'_1) = \pi_{t',t}(\Omega'_2)$ which contradicts the previously established fact that $\pi_{t',t}$ is a bijection for such t, t' and so proves the proposition. $\square$

4.4 Proof of Theorem 8

In this section, we will show Theorem 8. In fact, we will show a stronger result from which Theorem 8 is an immediate consequence.

Theorem 9. *Let Σ be a smooth closed connected hypersurface in $\mathbb{R}^4$ with $\lambda[\Sigma] \leq \Lambda_2$. If $\{\Gamma_t\}_{t\in[0,T]}$ is the level set flow with $\Gamma_0 = \Sigma$ and extinction time T, then, for all $t \in [0,T]$, Γ_t is connected and $n(t) \leq 2$. Moreover, if*

$$E^+ = W^+[0,T] \text{ and } E^- = W^-[0,T] \cup \left(\mathbb{R}^4 \times (T,\infty)\right),$$

then $E^\pm$ are both sets of locally finite perimeter in $\mathbb{R}^4 \times [0,\infty)$ and there are Brakke flows $\mathcal{K}^\pm$ so that

$$(\tau^\pm = \pm\left(\partial[E^\pm] + [W^\pm[0] \times \{0\}]\right), \mathcal{K}^\pm)$$

are both matching motions with initial condition $[\Sigma \times \{0\}]$. Finally,

$$\overline{\partial^* E^\pm} = \partial E^\pm$$

in $\mathbb{R}^4 \times (0,\infty)$.

Proof. First observe that we may assume $\lambda(\Sigma) < \Lambda_2$. Indeed, suppose that $\lambda(\Sigma) = \Lambda_2$ and consider, $\{\Sigma_t\}_{t\in[0,\delta]}$, the classical solution to MCF equation with $\Sigma_0 = \Sigma$. As Σ is closed, $\lambda(\Sigma) = F[\rho^{-1}(\Sigma - x)]$ for some $\rho > 0$ and $x \in \mathbb{R}^{n+1}$. Hence, by the Huisken monotonicity formula, either $\lambda[\Sigma_\delta] < \Lambda_2$ or $\Sigma = \rho\Upsilon + x$ where Υ is a closed self-shrinker. In the latter case, the theorem is immediate (as the flow will remain smooth until disappearing), while in the former, one can prove the result for Σ_δ and then use the fact that the flow was smooth to conclude it also for Σ.

As Σ is a closed connected hypersurface in $\mathbb{R}^4$, standard topological results, e.g., [33], imply that there is a connected bounded domain $E_0 \subset \mathbb{R}^4$ with $\partial E_0 = \Sigma$. Let $\mathbf{n}$ be the unit normal to Σ that points into E_0. As Σ is smooth, there is an $\epsilon > 0$ so for $|s| < \epsilon$

$$\Sigma_s = \{p + s\mathbf{n}(p) | p \in \Sigma\}$$

is a foliation of $T_\epsilon(\Sigma)$ by hypersurfaces . By shrinking ϵ, if needed, we can also ensure that $\lambda(\Sigma_s) < \Lambda_2$

for $|s| < \epsilon$. Pick a Lipschitz function $u_0 : \mathbb{R}^4 \to \mathbb{R}$ with the property that

1. $\{u_0 = s\} = \Sigma_s$ for $|s| < \epsilon$,

2. $\{u_0 \leq -\epsilon\}$ is the unbounded component of $\mathbb{R}^4 \backslash T_\epsilon(\Sigma)$; and

3. $\{u_0 \geq \epsilon\}$ is the bounded component of $\mathbb{R}^4 \backslash T_\epsilon(\Sigma)$.

Let u be the solution to 3.2.5 with initial data u_0. As such, if $\Gamma_t^s = \{x | u(t,x) = s\}$, then for $|s| < \epsilon$, $\{\Gamma_t^s\}_{t \geq 0}$ is the level set flow with $\Gamma_0^s = \Sigma_s$. For each $i \geq 1$, pick $s_{\pm i} \in (-\epsilon, \epsilon)$ so that $s_{-i} < s_{-i-1} < 0 < s_{i+1} < s_i$ and $\lim_{i \to \pm\infty} s_i = 0$. Let $E_0^i = \{u_0 > s_i\}$ and $E^i = \{u > s_i\}$. By [25, 12.11], one can choose the s_i so that for $i \neq 0$, there are Brakke flows $\mathcal{K}^i$ so that $\left(E_0^i, E^i, \mathcal{K}^i\right)$ are all strong canonical boundary motion.

By Proposition 4.3.1, each $\Gamma_t^i = \Gamma_t^{s_i} = \{u = s_i\}$ is connected and for $t \in [0, T_i)$, where T_i is the extinction time of the flow, divides $\mathbb{R}^4$ into two components $W_i^{\pm}[t]$ which satisfy $\Gamma_t^i = \partial W_i^{\pm}[t]$ and $W_i^+[t] = E_t^i = \{x | u(t,x) > s_i\}$. Consider the open sets

$$U^+[t] = \bigcup_{i=1}^{\infty} W_i^+[t] = \{x | u(x,t) > 0\} \text{ and } U^-[t] = \bigcup_{i=1}^{\infty} W_{-i}^-[t] = \{x | u(x,t) < 0\}.$$

As each $W^{\pm}[t]$ is connected and $U^{\pm}[t]$ is their nested union, it follows that both the $U^{\pm}[t]$ are also connected. Moreover, as

$$\Gamma_t = \{x | u(x,t) = 0\} = \mathbb{R}^4 \backslash \left(U^+[t] \cup U^-[t]\right),$$

$W^{\pm}[t] = U^{\pm}[t]$. For $i \geq 1$ let,

$$G_i[t] = \mathbb{R}^4 \backslash \left(W_i^+[t] \cup W_{-i}^-[t]\right) = \{x | s_{-i} \leq u(x,t) \leq s_i\}$$

and observe that each $G_i[t]$ is a compact set, $G_{i+1}[t] \subset G_i[t]$ and $\bigcap_{i=1}^{\infty} G_i[t] = \Gamma_t$. For $t \in [0,T]$, each $G_i[t]$ is connected. Indeed, T_{-i}, the extinction time of $\left\{\Gamma_t^{-i}\right\}_{t \geq 0}$ must satisfy $T_{-i} > T$ and so, when $t \leq T$, Γ_t^{-i} and $W_{-i}^{\pm}[t]$ are both non-empty and connected. In particular, there is exactly one component, $G_i^-[t]$, of $G_i[t]$ that contains $\Gamma_t^{-i} = \partial W_{-i}^{\pm}[t]$. Let $G_i^+[t] = G_i[t] \backslash G_i^-[t]$, so $G_i^+[t]$ is closed and disjoint from $G_i[t]$. Observe that $W_{-i}^-[t] \cup G_i^-[t]$ is a closed non-empty subset of $\overline{W_i^-[t]} = W_i^-[t] \cap \Gamma_t^i = \{u \leq s_i\}$ that is disjoint from $G_i^+[t]$. As $G_i^+[t]$ is also a closed subset of $\overline{W_i^-[t]}$, $\overline{W_i^-[t]} = W_{-i}^-[t] \cup G_i^-[t] \cup G_i^+[t]$ and the closure of a connected set is connected, $G_i^+[t] = \emptyset$, and so $G_i[t]$ is connected. As the nested intersection of compact connected sets is connected, it follows that Γ_t is connected and so we've proved the first part of the theorem.

To prove the second part of the theorem we begin observe that for $i \geq 1$, $E^i = W[0,T]$ is a set of finite perimeter while

$$F^{-i} = \{u < s_{-i}\} = \mathbb{R}^4 \times [0,\infty) \backslash \bar{E}^{-i},$$

is a set of locally finite perimeter. Moreover, there are matching motions

$$\left(\tau^i = \partial[E_i] + W_i^+[0]], \mathcal{K}^i\right) \text{ and } \left(\tau^{-i} = -\left(\partial[F^{-i}] + [W_{-i}^-[0]]\right), \mathcal{K}^{-i}\right)$$

with initial conditions $[\Sigma_{s_{\pm i}} \times \{0\}]$. As $\lambda(\Sigma_{s_{\pm i}}) < \Lambda_2 < 2$, [35, Theorem 3.4] implies that, up to passing to a subsequence, the two sequences of matching motions converge to matching motions $(\tau^+, \mathcal{K}^+)$ and $(\tau^-, \mathcal{K}^-)$ both with initial condition $[\Sigma \times \{0\}]$. It further follows from standard compactness results for sets of locally finite perimeter, that E^i converges as a set of finite perimeter to

$$E^+ = W^+[0,T] = \bigcup_{t\in[0,T]} U^+[t] = \{u > 0\}$$

which is also a set of finite perimeter, while F^{-i} converges as a set of locally finite perimeter to F^-. One readily verifies that

$$F^- = W^-[0,T] \cup \left(\mathbb{R}^4 \times (T,\infty)\right) = \left(\bigcup_{t\in[0,T]} U^-[t]\right) \cup \left(\mathbb{R}^4 \times (T,\infty)\right) = \{u < 0\}\,.$$

Set $E^- = F^-$ and observe that $\tau^\pm = \pm\left(\partial[E^\pm] + [W^\pm[0]]\right)$ follows from the continuity of the boundary operation.

It remains only to verify the claim about the reduced boundary. To that end observe that in $\mathbb{R}^4 \times (0,\infty)$

$$\overline{\partial^* E^+} \subset \partial E^+.$$

We now suppose that $(x,t) \in \partial E^+$ and $t > 0$. By definition, for any $r > 0$, $B_r(x,t) \cap E^+ \neq \emptyset$. In particular, for i sufficiently large $B_r(x,t) \cap W_i^+[0,T] \neq \emptyset$. As $x \in \Gamma_t$, we have $x \notin W_i^+[0,T]$ and so there is some point $(y_r, t_r) \in B_r(x,t) \cap \partial W_i^+[0,T]$. As $\left(E_0^i, E^i, \mathcal{K}^i\right)$ is a strong canonical boundary motion, it has only one compact singularity (at the terminal time $T_i < T$) and we can assume $t_r < T_i$. Hence, by Proposition 9 that $y_r \in \operatorname{spt}(\mu^i_{t_r})$ and so (y_r, t_r) has positive Gaussian density for $\mathcal{K}^i$. As $\mathcal{K}^i$ converges to $\mathcal{K}^+$, the upper semicontinuity of Gaussian density implies that (y,t) is a point of positive Gaussian density for $\mathcal{K}^+$. As $(\tau^+, \mathcal{K}^+)$ is a matching motion starting from

Σ and τ^+ is the reduced boundary of a set of finite perimeter, $(y,t) \in \overline{\partial^* E^+}$. That is, $\overline{\partial^* E} = \partial E^+$ in $\mathbb{R}^4 \times (0,\infty)$. Arguing in exactly the same way shows that $\overline{\partial^* E^-} = \partial E^-$ in $\mathbb{R}^4 \times (0,\infty)$ $\square$

Corollary 3. *Let Σ be a smooth closed connected hypersurface in $\mathbb{R}^4$ with $\lambda[\Sigma] \leq \Lambda_2$. If $\{\Gamma_t\}_{t\in[0,T]}$, the level set flow of Σ with extinction time T, is non-fattening, then there is a unique strong canonical boundary motion $(E_0, E, \mathcal{K})$, with $\partial E_0 = \Sigma$.*

4.5 A sharp entropy bound for forward clearing out

In this section apply Theorem 8 to prove Corollary 2.

Proof of Corollary 2. If the Corollary is not true, then there exist $C_i \to 0, \eta_i > 0, R_i > 0, 0 < \rho_i < \frac{R_i}{2C_i}$ satisfying $\frac{\eta_i}{C_i^3} \to 0$ and a sequence of non-fattening level set flows $\{M_{i,t}\}_{t\geq 0}$ with $M_{i,0}$, closed hypersurfaces with $\lambda(M_{i,0}) < \Lambda_2$, $M_{i,t} \neq \emptyset$ for $t \in (t_0, t_0 + R_i^2)$ and so that the flows reach the space-time point (x_0, t_0), but satisfy

$$\mathcal{H}^3(B_{\rho_i}(x_0) \cap M_{t_0 + C_i^2\rho_i^2}) < \eta_i \rho_i^3.$$

By Corollary 3, the $M_{i,t}$ agree with the slices of a strong canonical boundary motion $(E_{i,0}, E_i, \mathcal{K}_i = \{\mu_{i,t}\})$. In particular, by Proposition 9,

$$\mu_{i,t} = \mathcal{H}^3 \lfloor M_{i,t}$$

and so $\mu_{i,t}(B_{\rho_i}(x_0)) < \eta_i \rho_i^3$.

Rescale the flows to get a new flow $\tilde{K}_i = D_{\frac{1}{C_i\rho_i}}(\mathcal{K}_i - (x_0, t_0))$ and let $\{\tilde{M}_{i,t}\}$ be the corresponding rescaling of the level set flow $\{M_t\}$. By Brakke's compactness theorem [25, 7.1], up to passing to a subsequence, $\tilde{K}_i$ converges to a limit flow $\tilde{K} = \{\tilde{\mu}_t\}$, and moreover, by [35, Theorem 3.5], $(T_i, \mathcal{K}_i)$ converge to a matching motion $(\tilde{T}, \tilde{K})$.

We also have by rescaling

$$\tilde{\mu}_{i,1}\left(B_{\frac{1}{C_i}}(0)\right) < \frac{\eta_i}{(C_i)^3} \to 0$$

That is, $\tilde{\mu}_1(\mathbb{R}^4) = 0$ and so the limit flow $\tilde{K}$ must be extinct before $t = 1$. As $(\tilde{T}, \tilde{K})$ is a matching motion, this means that $\tilde{K}$ must develop a collapsed singularity at some $t_e \leq 1$. By the classification of singularities given in Proposition 8, this singularity has compact support. Hence, by Brakke's regularity theorem, for large enough i, the flow $\{\tilde{M}_{i,t}\}$ must develop a compact singularity at some time $\tilde{t}_i < 2$, and hence $\{M_{i,t}\}$ must develop compact singularity at some time $t_i < t_0 + 2C_i^2\rho_i^2 < t_0 + \frac{2R_i^2}{4} < t_0 + R_i^2$. Since $M_{i,t_0+R_i^2} \neq \emptyset$ and there is a compact singularity before the extinction time, there must be disconnection before time $t_0 + R_i^2$, contradicting Theorem 8. $\square$

5

Topological rigidity of compact self-shrinkers

5.1 Main result

In this chapter we show that at any rate when $n = 2$ they must be "topologically standard" in the following way:

Theorem 10. *Let $F : \Sigma_g \to \mathbb{R}^3$ be an embedded closed self shrinker of genus g. Then it is isotopic to the standard genus g surface in $\mathbb{R}^3$.*

As a corollary:

Corollary 4. *Let $F : T^2 \to \mathbb{R}^3$ be an embedded self shrinking torus. Then it is unknotted.*

We recall a knot is a closed embedded curve $\gamma : S^1 \to \mathbb{R}^3$. γ is unknotted if it is ambiently isotopic to the equator of the round 2-sphere of radius 1 in $\mathbb{R}^3$. Similarly, in this article we will say a hypersurface $F : T^2 \to \mathbb{R}^3$ is **unknotted** if it is ambiently isotopic to a tubular neighborhood of an unknotted curve γ.

Analogous results have been found by Lawson in [28] for minimal hypersurfaces in S^3 and Freedman, Frohman, Meeks, Yau, and others for different classes of minimal hypersurfaces in $\mathbb{R}^3$ - see [29], [20], [19].

In the conclusion of his paper Lawson gives (or at least restates) his famous conjecture (solved finally by Brendle in [8]) that the Clifford torus is the only embedded minimal torus in S^3, his result (in particular, that there are no minimal knotted tori in S^3) meant as a step towards this. Similarly, we feel inclined to optimistically conjecture:

Conjecture 1. *The only embedded self shrinking torus in* $\mathbb{R}^3$ *is Angenent's torus.*

This conjecture is not completely unfounded. Angenent used a "shooting method" to show provide a curve that, when rotated about an axis, gives a self-shrinker. This method suggests at least local isolatedness but the authors presently don't know how to prove this. We might also note (to be taken perhaps as evidence to the contrary) that Angenent in [1], the same paper where he constructs his famous embedded torus, constructs many immersed (not embedded) tori and Drugan and Kleene in [12] also constructed many immersed tori. Of course, there are many immersed CMC surfaces but only the spheres are embedded, so perhaps this evidence isn't so threatening.

In another direction, we recall that $\mathbb{R}^3$ with the Gaussian metric is Ric_f positive in the sense of Bakry and Emery so, given the great analogies between f-Ricci positivity and Ricci positivity in comparison geometry (see [36] for theorems along this line and a nice introduction to the subject) we might feel compelled to also suggest that in f-Ricci positive 3-manifold, for at least certain f, an unknottedness/rigidity theorem in the sense of the above holds. The authors presently haven't investigated this further.

5.2 Heegaard splitting of $\mathbb{S}^3$

In this section we give a review of Heegaard splitting of 3-manifolds and Waldhausen's results on the uniqueness of Heegaard splitting of 3-sphere. The main reference is [32].

A 3-dimensional handle body is a homeomorph of closed regular neighborhoods of a finite, connected graph.

Definition 7. *A Heegaard splitting is a pair* (M, F) *where* M *is a closed oriented three-manifold,* F *is an oriented closed surface embedded in* M*, and* $M \setminus F = V \cup W$ *is a disjoint union of handlebodies of genus* g*.* V *and* W *are glued together along the Heegaard surface* F*.*

Definition 8. *2 Heegaard splitting are said to be equivalent if they are the same up to an ambient isotopy that preserves the orientation of the Heegaard surface.*

The famous Waldhausen's theorem on the uniqueness of Heegaard splitting of S^3 states that

Theorem 11. *If* F *and* F' *are 2 Heegaard splitting of* $\mathbb{S}^3$ *of the same genus, then they are equivalent.*

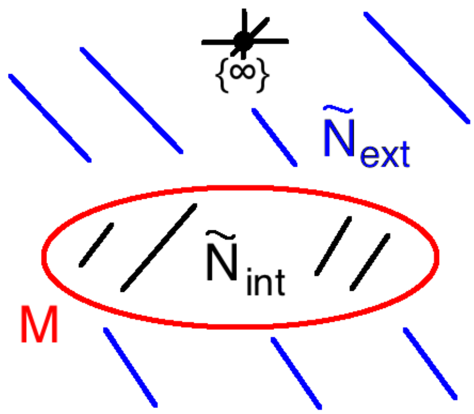

Figure 5.1:

5.3 Proof of the Theorem 10

Let's start by setting some notation: M will be a compact self shrinker of genus g. Denote by N_{int} and N_{ext} the interior and exterior components of M in $\mathbb{R}^3$, and $\widetilde{N_{int}}$ and $\widetilde{N_{ext}}$ by their corresponding sets by adding $\{\infty\}$ and thinking of S^3 as the one point compactification of $\mathbb{R}^3$ via stereographic projection (since M is compact, as sets note $\widetilde{N}_{int} = N_{int}$). The following diagram is what the reader should have in mind:

We will also need that the self-shrinker M is a minimal surface in the Gaussian metric $(\mathbb{R}^3, \frac{1}{(4\pi)} e^{\frac{-|x|^2}{4}} \delta_{ij})$, so as M is the boundary of both N_{int} and N_{ext} they both are manifolds with mean convex boundary.

In [28], Lawson proved the analogue of theorem 1 for minimal surfaces in S^3 by showing that minimal surfaces are Heegaard splitings of S^3; which is to say their interior and exterior components are Handlebodies. He then appealed to the theorem of Waldhausen 11. For the main geometric step in his argument, which is essentially lemma 2.1 below, Lawson uses the Frenkel argument [18] which crucially relies on the positivity of the Ricci curvature of S^3, which the Gaussian metric doesn't satisfy. Instead we apply a minimization argument of Frohman-Meeks [20]; some modifications similar to what's found in Brendle's paper [8] are necessary though since the metric decays to the zero form at infinity.

Lemma 9. *Let M be a connected compact self shrinker in $\mathbb{R}^3$ or equivalently a connected minimal surface in $(\mathbb{R}^3, G)$ where G is the Gaussian metric. Then the map*

$$\pi_1(\partial N_{ext}) \xrightarrow{i_*} \pi_1(N_{ext}) \tag{5.3.1}$$

induced by inclusion is surjective. The same is true for N_{int}.

In the following we state precisely the minimization lemma of Frohman and Meeks we mentioned above; in the statement an almost-complete Riemannian surface means namely a complete metric space with respect to the distance function induced by infimum of length curves joining 2 points. It's proof is sketched below in the course of proving surjectivity $\pi_1(\partial N_{ext}) \xrightarrow{i_*} \pi_1(N_{ext})$:

Lemma 5.3.1. *(Lemma 3.1 of [20])*

Suppose N is a connected, orientable, almost-complete Riemannian three-manifold with more than one boundary component. If ∂N has nonnegative mean curvature with respect to the outward pointing normal, then N contains a properly embedded, orientable, least-area minimal surface.

Before starting the proof, let's recall some basic facts about the length functional. For a path γ in a Riemannian manifold (N, μ), denote its length with respect to μ by $L_\mu(\gamma) = \int \mu(\dot{\gamma}, \dot{\gamma})dt$. This let's us induce a metric space topology on (N, μ) by defining the distance between two points p and q to be given by $\inf\limits_{\{\gamma|\gamma(0)=p,\gamma(1)=q\}} L_{\sigma\mu}(\gamma)$. We note that if, as positive definite matrices over each point, two metrics g_1, g_2 on N satisfy $cg_1 < g_2 < Cg_1$ for $c, C > 0$, then we have for their length functionals $L_{cg_1} < L_{g_2} < L_{Cg_1}$. Hence if as a metric space induced by the length functional (N, g_1) is complete, so is (N, g_2).

Proof. Since M is compact and in every compact ball there is some constant σ so that $\sigma\delta_{ij} < G < \delta_{ij}$, from the comments above the (N_{int}, G) is almost complete so we get from lemma 5.3.1 a stable self shrinker $\pi(\Sigma) \subset N_{int}$. From here we may proceed from claim 2.1 below onward to conclude.

For N_{ext} we follow closely the argument of Frohman and Meeks (but enough modifications are necessary that we don't just quote their lemma). Suppose $\pi_1(M) \to \pi_1(N_{ext})$ is not surjective, $M = \partial N_{ext}$. Then by elementary covering space theory $\partial \bar{N}_{ext}$ is not connected, where $\partial \bar{N}_{ext}$ is the boundary of the universal cover $\pi : \bar{N}_{ext} \to N_{ext}$. Denote by ∂_1, ∂_2 two (of perhaps many) connected components of $\partial \bar{N}_{ext}$.

We know $\partial N_{ext} = M$ is a minimal surface in $(N_{ext}, G = e^{-\frac{|x|^2}{4}}\delta_{ij})$, so ∂_1, ∂_2 as subsets of the lift of M are also minimal surfaces in $(\bar{N}_{ext}, \bar{G})$, where $\bar{G}$ is the lift of G and hence are mean convex. Denote $\gamma \subset \bar{N}_{ext}$ a smooth curve connecting ∂_1, ∂_2. At this point Frohman and Meeks find a stable minimal surface disjoint from the boundary but $(\bar{N}_{ext}, \bar{G})$ is not almost-complete so we can't do that yet, so we perturb the metric as in [8]. Let Ψ be a smooth bump function in $\mathbb{R}^3$ defined by

$$\begin{aligned} &\Psi(x) = 1, x \in B(0,1) \\ &\Psi(x) = 0, x \in \mathbb{R}^3 \setminus B(0,2) \end{aligned} \tag{5.3.2}$$

and

$$\Psi_k(x) = 1 - \Psi(\frac{x}{k}) \tag{5.3.3}$$

where $B(a,R)$ is the open ball of radius R centered at a. We use these functions to perturb the Gaussian metric G of N_{ext} and define for any $k \in \mathbb{N}$

$$G_k = (e^{-\frac{|x|^2}{4}} + \Psi_k)\delta_{ij} \tag{5.3.4}$$

We can choose $k_0 \in \mathbb{N}$ large enough so that

$$\begin{aligned} &M \subset B(0,k_0) \\ &\pi(\gamma) \subset B(0,k_0) \end{aligned} \tag{5.3.5}$$

Now for any $k > k_0, \partial_1, \partial_2$ are minimal surface in the lifted metric $(\bar{N}_{ext}, \bar{G}_k)$, moreover $(\bar{N}_{ext}, \bar{G}_k)$ is a complete Riemannian metric and an almost complete metric space (from what we discussed in the paragraph before the start of the proof) satisfying conditions of Lemma 5.3.1, so we get a properly embedded connected area minimizing surface $\Sigma_k \subset (\bar{N}_{ext}, \bar{G}_k)$. For the sake of completeness let's briefly explain how Σ_k is found:

Let $\Pi_1 \subset \Pi_2 \subset \ldots$ be a smooth compact exhaustion of ∂_1 with $p \in \Pi_1$. Recalling that the metric was perturbed away from M so the boundary in the universal cover is still mean convex, by standard existence and regularity theory for minimizers (completeness of space necessary) for each i we may find Σ_i properly embedded, orientable area minimizer with boundary $\partial\Pi_i$ as in the following picture:

Since each of the Σ_i are area minimizing we get local area bounds which lets us apply standard compactness and regularity theory to take a limit of the Σ_i to get an area minimizing surface we denote Σ_k. Since each of the Σ_i intersected the arc γ with odd intersection number, the limit surface Σ_k is nonempty and intersects with γ. To proceed we need the following sub-lemma:

Claim 5.3.2. *Σ_k is disjoint from $\partial\bar{N}_{ext}$.*

Proof. If not, then by maximum principle $\Sigma_k \subset \partial\bar{N}_{ext}$ and Σ_k agrees with one of the connected

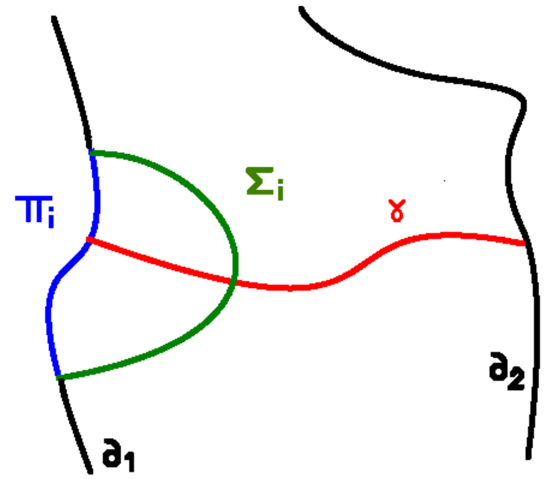

Figure 5.2:

components of $\partial\bar{N}_{ext}$, thus $\pi(\Sigma_k) = M$.

Since Σ_k is area minimizing in $(\bar{N}_{ext,}, \bar{G}_k)$, M is stable in (N_{ext}, G_k). But $k > k_0$ implies G and G_k agree in a neighborhood of M, meaning M is stable in (N_{ext}, G). Now recall from Colding-Minicozzi (namely theorem 4.14 in [10]) that if we take $L_M = \Delta_M + |A|^2 + \frac{1}{2} - \frac{1}{2}\nabla_{x^T}$, then for a self shrinker the second variation is given by

$$- \int_M e^{-\frac{|x|^2}{4}} fLf = \int_M e^{-\frac{|x|^2}{4}} (|\nabla^M f|^2) - |A|^2 f^2 - \frac{1}{2} f^2) \tag{5.3.6}$$

for test functions thatt vanish along the boundary of M. If M is closed and we plug in the test function $f = 1$ we have $L(f) > 0$ so the constant outward normal vector field on M will decrease area, implying that M is not stable in (N_{ext}, G), a contradiction.

□

Now $\Sigma_k \cap B(0, k_0) \neq \emptyset$ because they all intersect with γ. We now let let $k \to \infty$. The ambient manifolds $(\bar{N}_{ext}, \bar{G}_k) \to (\bar{N}_{ext}, \bar{G})$ and the stable minimal surfaces after passing to a subsequence (uniform ambient curvature bounds imply uniform local area bounds for area minimizers, which implies curvature (and higher order) bounds) converge to some surface $\Sigma_k \to \Sigma$ where $\Sigma \subset (\bar{N}_{ext}, \bar{G})$ is a non-empty stable minimal surface that is disjoint from $\partial\bar{N}_{ext}$. When projecting down to the base, $\pi(\Sigma) \subset (N_{ext}, G)$ is an embedded minimal surface that is disjoint from M, namely it's a self-shrinker in $\mathbb{R}^3$ and lies in N_{ext}.

As disjoint self-shrinkers, we have the distance between M and $\pi(\Sigma)$ becomes 0 at $t = 0$ (equation (1.1) is a normalized equation for self shrinkers that implies solutions are extinct at $t = 1$ under their flow), but this is violating the maximum principle which implies that the distance of two disjoint submanifold, if one of them is compact (which M is), is non-decreasing under the flow - this gives a contradiction. Thus the boundary of $\bar{N}_{ext}$ is connected and the statement is true. $\square$

Of course, in the one point compactification, since M is compact we see as sets $\widetilde{N}_{int} = N_{int}$. N_{ext} is related to $\widetilde{N}_{ext}$ by way of the following observation:

Lemma 5.3.3. *Suppose that $N \subset \mathbb{R}^3$ is a 3 manifold with boundary that is the exterior of a closed compact surface M, and let $\widetilde{N}$ be the compactification of N induced by the one point compactification of $\mathbb{R}^3$ by adding a point. Then the induced map on $\pi(N) \to \pi(\widetilde{N})$ by inclusion is surjective.*

Proof. This is because N contains a neighborhood of spatial infinity because K is compact, and because the 2-sphere is simply connected. $\square$

Thus, arguing as in the proof of Theorem 3 in [28] one can show that M is a Heegaard surface and so is isotopic to the standard surface of genus g in S^3 by Waldhausen's theorem 11, so any two compact self shrinkers of genus g in $\mathbb{R}^3$ are ambiently isotopic to each other in the one point compactification of $\mathbb{R}^3$. Of course, we may arrange this isotopy to avoid any particular point on S^3 so that (by stereographic projection say) the surfaces are isotopic in $\mathbb{R}^3$, giving theorem 10.

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Curriculum Vitae

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